POW#1: An Integral Sequence

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$Proposition. If\ a\ finite\ sequence\ \{a_1, \dots, a_n\}\ in\ \mathbb{R}\ satisfies\ the\ following, then\ a_i \in \mathbb{Z}\ (i = 1, \dots, n)$

$$\sum_{i=1}^{m} a_i^3 = \left(\sum_{i=1}^{m} a_i\right)^2\ (m = 1, \dots, n)\ \cdots (1)$$

$We\ prove\ by\ induction\ on\ n.$
$For\ n = 1, a_1\ is\ either\ 0\ or\ 1, thus\ the\ proposition\ stands\ true.$
$Now\ assume\ that\ the\ proposition\ holds\ for\ n \leq k. Consider\ the\ case\ in\ which\ n = k + 1.$
$Let\ J = \{i: a_i = 0\}. If\ J \neq \emptyset, then\ the\ problem\ reduces\ to\ the\ case\ where\ n \leq k\ because$
$at\ least\ one\ a_i\ is\ negligible\ in\ eq. (1).$
$Now\ assume\ J = \emptyset. The\ fact\ that\ \{1, \dots, k + 1\}\ is\ a\ valid\ sequnce\ for\ \{a_n\}\ can\ be\ shown\ easily.$
$Since\ J = \emptyset, a_1 = 1. For\ a\ valid\ sequence\ \{a_n\}\ different\ from\ \{1, \dots, k + 1\}, let\ j = \min\{i: a_i \neq i\}.$
$Clearly, 2 \leq j \leq k + 1. \sum_{i=1}^{j-1} i^3 + a_j^3 = \left(\sum_{i=1}^{j-1} i + a_j\right)^2\ implies\ a_j^2 - a_j - k(k + 1) = 0, and$
$thus\ a_j = -k. Since\ k^3 + (-k)^3 = 0\ and\ k + (-k) = 0, a_j cancels\ out\ a_{j-1}\ in\ eq. (1).$
$Therefore, the\ problem\ is\ reduced\ to\ the\ case\ where\ n \leq k - 1.$
$By\ the\ induction\ hypothesis, the\ proposition\ holds\ for\ every\ positive\ integer\ n.$