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Let $\mathbb{F}_2$ denote the finite field of two elements. For each $i = 1, 2, \dots, n$ let $\mathbf{v}_i \in \mathbb{F}_2^n$ be a vector where its k^{th} element is defined as

$$(\mathbf{v}_i)_k = \begin{cases} 1 & \text{if switch } s_i \text{ controls } \ell_k, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\mathbf{v}_i$ indicates which light bulbs are controlled by switch s_i.

Let us express the states of the light bulbs as a vector in $\mathbb{F}_2^n$, where 1 corresponds to a light bulb being turned on and 0 corresponds to it being turned off. Then by the rules of arithmetic in $\mathbb{F}_2$, we see that flipping the switch s_i is the equivalent of adding $\mathbf{v}_i$ to the state vector. In this point of view, we claim the following:

Claim 1. *There exists a subset $\{\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_m}\}$ of $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ whose sum of all the elements is equal to the all-ones vector $\mathbb{1} := (1, 1, \dots, 1)$.*

Note that if the claim holds, then by flipping the switches $\mathbf{s}_{i_1}, \dots, \mathbf{s}_{i_m}$ once each, we will end up turning all the lights on.

Now we prove the claim. Define a matrix

$$\mathbf{V} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} \in \mathbb{F}_2^{n \times n}$$

then for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{F}_2^n$ we have $\mathbf{V}\mathbf{x} = x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n$, so it suffices to show the existence of $\mathbf{x}$ such that $\mathbf{V}\mathbf{x} = \mathbb{1}$. Recall that we have the restriction that if s_i flips the status of ℓ_j then s_j flips the status of ℓ_i, which asserts that $(\mathbf{v}_i)_j = (\mathbf{v}_j)_i \Leftrightarrow (\mathbf{v}_i)_j + (\mathbf{v}_j)_i = 0$, or in other words that $\mathbf{V}$ is symmetric. Hence it holds that

$$\begin{aligned} \mathbf{x}^\top \mathbf{V}\mathbf{x} &= \sum_{j=1}^n \sum_{i=1}^n x_j (\mathbf{v}_i)_j x_i \\ &= \sum_{i=1}^n (\mathbf{v}_i)_i x_i^2 + \sum_{1 \leqslant i < j \leqslant n} \Big((\mathbf{v}_i)_j + (\mathbf{v}_j)_i \Big) x_i x_j \\ &= \sum_{i=1}^n (\mathbf{v}_i)_i x_i \\ &= \mathbb{1}^\top \mathbf{x} \end{aligned}$$

where in the third line we use the fact that $x^2 = x$ for any $x \in \mathbb{F}_2$, and in the fourth line we use the given condition that i^{th} switch flips the status of the i^{th} light hence $(\mathbf{v}_i)_i = 1$ for all $i = 1, \dots, n$. In particular, if $\mathbf{x} \in \ker(\mathbf{V})$, then $\mathbf{V}\mathbf{x} = \mathbf{0}$ so we must have $\mathbb{1} \perp \mathbf{x}$. That is,

$$\ker(\mathbf{V}) \subset (\operatorname{span}\{\mathbb{1}\})^\perp.$$

Taking the orthogonal complements, we obtain

$$\mathbb{1} \in \operatorname{span}\{\mathbb{1}\} \subset \operatorname{row\,sp}(\mathbf{V}).$$

But $\mathbf{V}$ is symmetric, so its row space is equal to its column space. Therefore, $\mathbb{1} \in \operatorname{col\,sp}(\mathbf{V})$, which means that there exists some $\mathbf{x} \in \mathbb{F}_2^n$ such that $\mathbf{V}\mathbf{x} = \mathbb{1}$. The proof is now complete.