KAIST Math Problem of the Week

2022-23 The number of eigenvalues of 8 by 8 matrices

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1 Problem

Let A be an 8 by 8 integral unimodular matrix. Moreover, assume that for each $x \in \mathbb{Z}^8$, we have $x^T Ax$ is even. What is the possible number of positive eigenvalues for A?

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Answer : A can have any number of positive eigenvalues.
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2 Solution

By taking $x = e_i$ and $x = e_i + e_j$ for $A = (a_{ij})$,

$$a_{ii} \quad \text{and} \quad a_{ij} + a_{ji} \quad \text{are even for all} \quad i, j = 1, ..., 8. \tag{1}$$

(1) is a sufficient condition for each $x \in \mathbb{Z}^8, x^T Ax$ is even. We will construct 8 by 8 unimodular matrices $A_i (0 \leq i \leq 8)$ satisfying (1) such that each A_i has exactly i-many positive eigenvalues. To do so, recall that

Lemma 1. *(Inverse of block matrix) If $A = (a_{ij})$ is a $(m+n)$ by $(m+n)$ block matrix given by*

$$a_{ij} = \begin{cases} b_{ij} & \text{if} \quad 1 \leq i, j \leq m \\ c_{kl} & \text{if} \quad m+1 \leq i, j \leq m+n, \ i = k+m, j = l+m \\ 0 & \text{otherwise}, \end{cases}$$

where $B = (b_{ij})$ is a m by m matrix and $C = (c_{kl})$ is a n by n matrix. Denote $A =$ Block(B, C) Then A is invertible if and only if both B and C are invertible and A^{-1} is given by $A^{-1} =$ Block(B^{-1}, C^{-1}).

The proof of the above lemma is straightforward. Next, consider the following 2 by 2 matrices;

$$B_0 = \begin{bmatrix} -2 & 3 \\ 1 & -2 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \tag{2}$$

$$C_2 = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}, D_2 = \begin{bmatrix} 6 & 7 \\ 5 & 6 \end{bmatrix}, E_2 = \begin{bmatrix} 8 & 9 \\ 7 & 8 \end{bmatrix} \tag{3}$$

B_i is unimodular and has i-many positive eigenvalues for $i = 0, 1, 2$. Also, C_2, D_2, and E_2 are unimodular and has 2 positive eigenvalues, respectively. No two matrices have common eigenvalues. Now it is time to end the proof; Define 8 by 8 matrices, with abusing the notation of "Block(,)",

$$\begin{aligned} A_0 &= \text{Block}(B_0, B_0, B_0, B_0) \\ A_1 &= \text{Block}(B_1, B_0, B_0, B_0) \\ A_2 &= \text{Block}(B_2, B_0, B_0, B_0) \\ A_3 &= \text{Block}(B_2, B_1, B_0, B_0) \\ A_4 &= \text{Block}(B_2, C_2, B_0, B_0) \\ A_5 &= \text{Block}(B_2, C_2, B_1, B_0) \\ A_6 &= \text{Block}(B_2, C_2, D_2, B_0) \\ A_7 &= \text{Block}(B_2, C_2, D_2, B_1) \\ A_8 &= \text{Block}(B_2, C_2, D_2, E_2). \end{aligned}$$

Immediately, $A_i's$ are 8 by 8 integral unimodular matrices such that $x^T A_i x$ is even for each $x \in \mathbb{Z}^8$, and A_i has $i-$many positive eigenvalues for $i = 0, ..., 8.\square$