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Let us define a new sequence $b_n = 1 + \sum_{k=1}^{n} a_k^2$. Notice that if $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of integers, then so is $\{b_n\}_{n\in\mathbb{N}}$. The given formula for $\{a_n\}_{n\in\mathbb{N}}$ asserts that $na_{n+1} = b_n$, hence $\{b_n\}_{n\in\mathbb{N}}$ satisfies the recurrence

$$b_{n+1} = b_n + a_{n+1}^2 = b_n + \frac{b_n^2}{n^2}.$$

This shows that, for $\{b_n\}_{n\in\mathbb{N}}$ to be a sequence of integers, each b_n should be divisible by n. Now fix any prime p, and consider the sequence $\{b_n\}_{n\in\mathbb{N}}$ modulo p. Then for $n < p$, from the recurrence relation derived above we should have

$$b_{n+1} \equiv b_n + b_n^2 n^{-2} \pmod{p} \tag{$*$}$$

which is well-defined as the multiplicative inverse of n modulo p always exists, and

$$b_p \equiv 0 \pmod{p}.$$

However, a direct computation using $(*)$ shows that $b_{43} \not\equiv 0 \pmod{43}$. Therefore, there exists some $n \geqslant 1$ such that a_n is *not* an integer.

The counterexample $p = 43$ was found using the following Python 3 code.

```python
primes = []
p = 1
while True:
    p += 1

    # find the next prime using sieve of Eratosthenes
    next_prime_found = False
    while not next_prime_found:
        composite = False
        for prev_p in primes:
            if prev_p ** 2 > p:
                break
            if p % prev_p == 0:
                composite = True
                break
        if composite:
            p += 1
        else:
            primes.append(p)
            break

    # test if p divides b_p using the recurrence relation
    bn = 2
    for n in range(1, p):
        bn = bn + pow(bn * pow(n, -1, p), 2, p)
    if bn % p != 0:
        print(p)
        break
```