Roughly, the following proposition gives an answer:

**Proposition 1.** The unimodular lattice of dimension 4, namely $\mathbb{Z}^4$, is unique up to isomorphism which preserves norms and inner products.

Let $A$ be a symmetric, positive-definite, unimodular matrix. There exists a decomposition $A = B^T B$ where $B$ is a real matrix.

Since $A$ is a real symmetric matrix, it has a diagonalization $A = Q^{-1} D Q$ where $Q$ is (real) unitary and $D$ is diagonal. Since $A$ is positive definite, every diagonal entry of $D$ is positive, so one can define $D^{1/2}$. Taking $B = D^{1/2} Q$, one has $B^T = Q^T D^{1/2} = Q^{-1} D^{1/2}$ and $B^T B = Q^{-1} D Q = A$.

Let $\{v_1, v_2, v_3, v_4\}$ be columns of $B$. They can be realized as elements of the real vector space $V = \mathbb{R}^4$ with the usual inner product $\langle v, w \rangle = v^T w$. Then, their Gram matrix $(\langle v_i, v_j \rangle)$ is exactly equal to $A$ from their construction.

Define a **fundamental mesh** $\Phi$ of $V$ by

$$
\Phi = \left\{ \sum_{i=1}^{4} x_i v_i : x_i \in \mathbb{R}, 0 \leq x_i < 1 \right\}.
$$

Seeing $V$ as the Euclidean space, we can assign the notion of volume to subsets of $V$. In this case, the volume of $\Phi$ is equal to 1, which is the absolute value of determinant of the matrix $B$ generated by the basis $\{v_i\}$. One has the following invariant notion:

$$
\langle \langle v_i, v_j \rangle \rangle = B^T B
$$

hence $\text{vol}(\Phi) = |\text{det}(\langle v_i, v_j \rangle)|^{1/2}$.

Let $\Lambda = \bigoplus_{i=1}^{4} \mathbb{Z} v_i$ be the integral span of $v_i$; usually this is called a **lattice** in $V$. If every inner product $\langle v, w \rangle$ is an integer for $v, w \in \Lambda$, $\Lambda$ is called **integral**. Note that $\Lambda$ is integral as its Gram matrix is integral. Further for integral lattice $\Lambda$, if the Gram matrix of a basis is of determinant 1, $\Lambda$ is called **unimodular**. Conclude that one can correspond a matrix from $S$ to an unimodular lattice in $\mathbb{R}^4$ using above procedure.

Let $X = \{ v \in V : \langle v, v \rangle \leq r^2 \}$ be the ball centered at the origin. Set $r = \frac{7}{5}$ (so that $\langle v, v \rangle < 2$ for $v \in X$) and claim that $X$ contains a nonzero point of $\Lambda$. 

1
Let $\frac{1}{2}X$ be the set of elements of $X$ multiplied by $\frac{1}{2}$. If there are two different points $v, w \in \Lambda$ such that $(v + \frac{1}{2}X) \cap (w + \frac{1}{2}X)$ is nonempty, then there exists $x_1, x_2 \in X$ such that $\frac{1}{2}x_1 + v = \frac{1}{2}x_2 + w$ hence $v - w = \frac{1}{2}(x_1 - x_2)$. Observe that $\frac{1}{2}(x_1 - x_2)$ is the center of the segment having $x_1, -x_2$ as ends. This point is contained in $X$ by following properties of $X$:

- $X$ is central symmetric: if $x_2 \in X$, then $-x_2 \in X$.
- $X$ is convex: for $x_1, x_2 \in X$ and $t \in [0, 1]$, $tx_1 + (1-t)x_2$ belongs to $X$. One may prove this in formal way, but one can easily get convinced recalling that a ball in the Euclidean space is convex.

Hence we have a nonzero element $v - w \in X \cap \Lambda$.

On contrary, assume that there is no such pair. Every $v + \frac{1}{2}X$ is disjoint to each other, so the same hold for the intersections with $\Phi$. Hence we have

$$\text{vol}(\Phi) \geq \sum_{v \in \Lambda} \text{vol}(\Phi \cap (v + \frac{1}{2}X)).$$

Translating $\Phi \cap (v + \frac{1}{2}X)$ by $-v$, it has the same volume as $(\Phi - v) \cap \frac{1}{2}X$. Traversing $v \in \Lambda$, fundamental meshes cover all the space, therefore one has

$$\text{vol}(\Phi) \geq \sum_{v \in \Lambda} \text{vol}((\Phi - v) \cap \frac{1}{2}X) = \text{vol}(\frac{1}{2}X) = \frac{1}{2^d} \text{vol}(X).$$

Note that the volume of a ball with radius $r$ in 4 dimensional Euclidean space is $\frac{1}{2}\pi^2 r^4$. Since $\frac{1}{2^d} \text{vol}(X) = \frac{20000}{79299} > 1 = \text{vol}(\Phi)$, this case does not happen.

This result is known as Minkowski’s lattice point theorem. (\cite{2}, theorem 4.4) One can apply the same with the hypothesis that $X$ is central symmetric, convex and $\text{vol}(\Phi) < \frac{1}{2^d} \text{vol}(X)$.

From above, deduce that $X$ contains a nonzero point of $\Lambda$. In particular, $X$ is the set of vectors with squared norm at most $3/2$. As $\Lambda$ is an integral lattice, deduce that there exists a nonzero point $x_1 \in \Lambda$ such that $\langle x_1, x_1 \rangle = 1$.

Let $\Lambda_1$ be the set of elements of $\Lambda$ that is orthogonal to $x_1$. Then $\Lambda$ is the direct sum of $\mathbb{Z}x_1$ and $\Lambda_1$. For $v \in \Lambda$, one has $v = \langle v, x_1 \rangle x_1 + (v - \langle v, x_1 \rangle x_1)$ with $(v, x_1) x_1 \in \mathbb{Z}x_1$, $v - \langle v, x_1 \rangle x_1 \in \Lambda_1$.

Let $V_1$ be the orthogonal complement of $x_1$. $\Lambda_1$ is the intersection of $V_1$ and $\Lambda$, and is integrally spanned by $v_1 - \langle v_1, x_1 \rangle x_1$. Choose a new basis $v_2', v_3', v_4'$ for $\Lambda_1$. Then $x_1, v_2', v_3', v_4'$ spans $\Lambda$. Consider the fundamental mesh of $\Lambda_1$. Its volume is the square root of the determinant of $\langle v_i', v_j' \rangle$.

Note that the volume of fundamental mesh of lattice is invariant under the basis change. Since the norm of $x_1$ is 1, $v_2', v_3', v_4'$ form a basis of $\Lambda_1$ in 3 dimensional vector space such that $|\det(\langle v_i', v_j' \rangle)| = 1$. The basis change from $\{v_i\}$ to $x_1$ and $\{v_i'\}$ can be done by multiplying some integral unimodular matrix. Note that for a unimodular lattice $\Lambda$, if there exists an element $x$ of norm 1 then $\Lambda$ is the direct sum of $\mathbb{Z}x$ and its complement.

In 3 dimensional space $V_1$, one proceed with the similar argument. Choose a ball $X$ with radius $\frac{1}{5}$, then $\frac{1}{2^d} \text{vol}(X) = \frac{3441}{79299} > 1$ hence there exists a nonzero
vector $x_2$ in $\Lambda_1$ such that $\langle x_2, x_2 \rangle = 1$ by Minkowski’s lattice point theorem. Let $V_2$ be the orthogonal complement of $x_1, x_2$, and $\Lambda_2$ be the set of points in $\Lambda$ that is orthogonal to $x_1, x_2$. $\Lambda_2$ is a unimodular lattice in $V_2$, and the basis change is done by multiplying an integral unimodular matrix, and so on. Note that for 2 dimensional unimodular lattice one can apply this Minkowski’s theorem by observing for a ball $X$ with $r = \frac{7}{2}$ that $\frac{1}{2\pi} \text{vol}(X) = \frac{49}{100} > 1$. Deduce that $\Lambda = \bigoplus_{i=1}^4 \mathbb{Z} x_i$, where $\{x_i\}$ is orthonormal.

One can find an integral basis change from columns of $B$ to an orthonormal basis $\{x_i\}$. The integral basis change is of the form of integral combination of columns of $B$: there exists an integral unimodular matrix $P$ such that the columns of $BP$ are $x_i$’s. This implies that $BP$ is an orthogonal matrix so that $P^T B^T BP = P^T AP = I$. Deduce that $A \sim I$ for any $A \in S$, therefore $S/\sim$ is the singleton set. This result also can prove proposition 1.

Note that we cannot use Minkowski’s theorem to find a vector of norm 1 for dimension $n \geq 5$ in this way because the volume of $n$-ball is not large enough. Despite of this, there is some non-elementary result that the minimum norm of an odd unimodular lattice of dimension $n$ is at most $1 + \lfloor \frac{n}{8} \rfloor$. (3, corollary 7.10) In fact, this result requires using theta function on lattice and properties of modular forms. Further, if unimodular lattice is even, then its dimension is divisible by 8. If one accepts these as facts, we can derive similar results up to dimension 7.

References

