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4 by 4 symmetric integral matrices

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Roughly, the following proposition gives an answer:

Proposition 1. *The unimodular lattice of dimension 4, namely $\mathbb{Z}^4$, is unique up to isomorphism which preserves norms and inner products.*

Let A be a symmetric, positive-definite, unimodular matrix. There exists a decomposition $A = B^T B$ where B is a real matrix.

Since A is a real symmetric matrix, it has a diagonalization $A = Q^{-1}DQ$ where Q is (real) unitary and D is diagonal. Since A is positive definite, every diagonal entry of D is positive, so one can define $D^{1/2}$. Taking $B = D^{1/2}Q$, one has $B^T = Q^T D^{1/2} = Q^{-1}D^{1/2}$ and $B^T B = Q^{-1}DQ = A$.

Let $\{v_1, v_2, v_3, v_4\}$ be columns of B. They can be realized as elements of the real vector space $V = \mathbb{R}^4$ with the usual inner product $\langle v, w\rangle = v^T w$. Then, their Gram matrix $(\langle v_i, v_j\rangle)$ is exactly equal to A from their construction.

Define a *fundamental mesh* Φ of V by

$$\Phi = \left\{ \sum_{i=1}^{4} x_i v_i : x_i \in \mathbb{R}, 0 \le x_i < 1 \right\}.$$

Seeing V as the Euclidean space, we can assign the notion of volume to subsets of V. In this case, the volume of Φ is equal to 1, which is the absolute value of determinant of the matrix B generated by the basis $\{v_i\}$. One has the following invariant notion:

$$(\langle v_i, v_j\rangle) = B^T B$$

hence $\mathrm{vol}(\Phi) = |\det(\langle v_i, v_j\rangle)|^{1/2}$.

Let $\Lambda = \bigoplus_{i=1}^4 \mathbb{Z}v_i$ be the integral span of v_i; usually this is called a *lattice* in V. If every inner product $\langle v, w\rangle$ is an integer for $v, w \in \Lambda$, Λ is called *integral.* Note that Λ is integral as its Gram matrix is integral. Further for integral lattice Λ, if the Gram matrix of a basis is of determinant 1, Λ is called *unimodular.* Conclude that one can correpond a matrix from S to an unimodular lattice in $\mathbb{R}^4$ using above procedure.

Let $X = \{v \in V : \langle v, v\rangle \le r^2\}$ be the ball centered at the origin. Set $r = \frac{7}{5}$ (so that $\langle v, v\rangle < 2$ for $v \in X$) and claim that X contains a nonzero point of Λ.

Let $\frac{1}{2}X$ be the set of elements of X multiplied by $\frac{1}{2}$. If there are two different points $v, w \in \Lambda$ such that $(v + \frac{1}{2}X) \cap (w + \frac{1}{2}X)$ is nonempty, then there exists $x_1, x_2 \in X$ such that $\frac{1}{2}x_1 + v = \frac{1}{2}x_2 + w$ hence $v - w = \frac{1}{2}(x_1 - x_2)$. Observe that $\frac{1}{2}(x_1 - x_2)$ is the center of the segment having $x_1, -x_2$ as ends. This point is contained in X by following properties of X:

- X is central symmetric: if $x_2 \in X$, then $-x_2 \in X$.
- X is convex: for $x_1, x_2 \in X$ and $t \in [0, 1]$, $tx_1 + (1-t)x_2$ belongs to X. One may prove this in formal way, but one can easily get convinced recalling that a ball in the Euclidean space is convex.

Hence we have a nonzero element $v - w \in X \cap \Lambda$.

On contrary, assume that there is no such pair. Every $v + \frac{1}{2}X$ is disjoint to each other, so the same hold for the intersections with Φ. Hence we have

$$\mathrm{vol}(\Phi) \geq \sum_{v \in \Lambda} \mathrm{vol}(\Phi \cap (v + \frac{1}{2}X)).$$

Translating $\Phi \cap (v + \frac{1}{2}X)$ by $-v$, it has the same volume as $(\Phi - v) \cap \frac{1}{2}X$. Traversing $v \in \Lambda$, fundamental meshs cover all the space, therefore one has

$$\mathrm{vol}(\Phi) \geq \sum_{v \in \Lambda} \mathrm{vol}((\Phi - v) \cap \frac{1}{2}X) = \mathrm{vol}(\frac{1}{2}X) = \frac{1}{2^4}\,\mathrm{vol}(X).$$

Note that the volume of a ball with radius r in 4 dimensional Euclidean space is $\frac{1}{2}\pi^2 r^4$. Since $\frac{1}{2^4}\,\mathrm{vol}(X) = \frac{2401\pi^2}{20000} > 1 = \mathrm{vol}(\Phi)$, this case does not happen.

This result is known as Minkowski's lattice point theorem. ([2], theorem 4.4) One can apply the same with the hypothesis that X is central symmetric, convex and $\mathrm{vol}(\Phi) < \frac{1}{2^n}\,\mathrm{vol}(X)$.

From above, deduce that X contains a nonzero point of Λ. In particular, X is the set of vectors with squared norm at most $3/2$. As Λ is an integral lattice, deduce that there exists a nonzero point $x_1 \in \Lambda$ such that $\langle x_1, x_1 \rangle = 1$.

Let Λ_1 be the set of elements of Λ that is orthogonal to x_1. Then Λ is the direct sum of $\mathbb{Z}x_1$ and Λ_1: For $v \in \Lambda$, one has $v = \langle v, x_1 \rangle x_1 + (v - \langle v, x_1 \rangle x_1)$ with $\langle v, x_1 \rangle x_1 \in \mathbb{Z}x_1$, $v - \langle v, x_1 \rangle x_1 \in \Lambda_1$.

Let V_1 be the orthogonal complement of x_1. Λ_1 is the intersection of V_1 and Λ, and is integrally spanned by $v_i - \langle v_i, x_1 \rangle x_1$. Choose a new basis v_2', v_3', v_4' for Λ_1. Then x_1, v_2', v_3', v_4' spans Λ. Consider the fundamental mesh of Λ_1. Its volume is the square root of the determinant of $(\langle v_i', v_j' \rangle)$.

Note that the volume of fundamental mesh of lattice is invariant under the basis change. Since the norm of x_1 is 1, v_2', v_3', v_4' form a basis of Λ_1 in 3 dimensional vector space such that $\left|\det(\langle v_i', v_j' \rangle)\right| = 1$. The basis change from $\{v_i\}$ to x_1 and $\{v_i'\}$ can be done by multiplying some integral unimodular matrix. Note that for a unimodular lattice Λ, if there exists an element x of norm 1 then Λ is the direct sum of $\mathbb{Z}x$ and its complement.

In 3 dimensional space V_1, one proceed with the similar argument. Choose a ball X with radius $\frac{7}{5}$, then $\frac{1}{2^3}\,\mathrm{vol}(X) = \frac{343\pi}{750} > 1$ hence there exists a nonzero vector x_2 in Λ_1 such that $\langle x_2, x_2 \rangle = 1$ by Minkowski's lattice point theorem. Let V_2 be the orthogonal complement of x_1, x_2, and Λ_2 be the set of points in Λ that is orthogonal to x_1, x_2. Λ_2 is a unimodular lattice in V_2, and the basis change is done by multiplying an integral unimodular matrix, and so on. Note that for 2 dimensional unimodular lattice one can apply this Minkowski's theorem by observing for a ball X with $r = \frac{7}{5}$ that $\frac{1}{2^2}\operatorname{vol}(X) = \frac{49\pi}{100} > 1$. Deduce that $\Lambda = \bigoplus_{i=1}^{4} \mathbb{Z}x_i$, where $\{x_i\}$ is orthonormal.

One can find an integral basis change from columns of B to an orthonormal basis $\{x_i\}$. The integral basis change is of the form of integral combination of columns of B: there exists an integral unimodular matrix P such that the columns of BP are x_i's. This implies that BP is an orthogonal matrix so that $P^T B^T BP = P^T AP = I$. Deduce that $A \sim I$ for any $A \in S$, therefore $S/\sim$ is the singleton set. This result also can prove proposition 1.

Note that we cannot use Minkowski's theorem to find a vector of norm 1 for dimension $n \geq 5$ in this way because the volume of n-ball is not large enough. Despite of this, there is some non-elementary result that the minimum norm of an *odd* unimodular lattice of dimension n is at most $1 + \lfloor \frac{n}{8} \rfloor$. ([3], corollary 7.10) In fact, this result requires using theta function on lattice and properties of modular forms. Further, if unimodular lattice is *even*, then its dimension is divisible by 8. If one accepts these as facts, we can derive similar results up to dimension 7.

References

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