KAIST Math Problem of the Week

2022-21 A determinant of greatest common divisors

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1 Problem

Let $\phi(x)$ be the Euler's totient function. Let $S = \{a_1, ..., a_n\}$ be a set of positive integers such that for any a_i, all of its positive divisors are also in S. Let A be the matrix with entries $A_{i,j} = \gcd(a_i, a_j)$ being the greatest common divisors of a_i and a_j. Prove that $\det(A) = \prod_{i=1}^{n} \phi(a_i)$.

2 Solution

By interchanging a_i and a_j for $i \neq j$, two rows and columns of A are interchanged. So the determinant of A is invariant under permutations and we may assume $a_1 < a_2 < ... < a_n$. Consider a lower triangular matrix $L = (L_{ij})$ defined by

$$L_{ij} = \begin{cases} \sqrt{\phi(a_j)} & \text{if} \quad a_j | a_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$A = LL^T, \tag{1}$$

$$\det(A) = (\det(L))^2 = \prod_{i=1}^{n} \phi(a_i). \tag{2}$$

To see (1), compute the (i, j) entry of LL^T

$$\sum_{k=1}^{n} L_{ik} L_{jk} = \sum_{a_k | a_i, a_k | a_j} \sqrt{\phi(a_k)} \sqrt{\phi(a_k)} = \sum_{a_k | \gcd(a_i, a_j)} \phi(a_k) = \gcd(a_i, a_j). \tag{3}$$

The identity

$$\sum_{d|n} \phi(d) = n \tag{4}$$

is used. One way to prove this is to consider $n-$fractions $\frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n}$. These fractions can be written in the form $\frac{c}{d}$ where $d|n$ and $\gcd(c, d) = 1$. By double counting, (4) is now proved and the problem is solved. □

References

[1] Z. Li The determinants of GCD matrices Linear Algebra Appl., 134 (1990), pp. 137-143