POW 2022-19
Inequality for Twice Differentiable Functions

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$Lemma.$ $The\ following\ holds\ for\ any\ differentiable\ function\ f: \mathbb{R} \to \mathbb{R}$

$$If\ f'(x) \geq d\ for\ x \in (a,b), then\ f(x) - f(a) \geq d(x-a)\ for\ x \in (a,b)$$

$$pf.\ f(x) - f(a) = \int_a^x f'(t)dt \geq \int_a^x d\ dt = f(x-a)$$

$Let\ g(x) = \left[\int_0^x f(t)dt\right]^2 - \int_0^x [f(t)]^3 dt\ \ for\ x \in \mathbb{R}.\ Clearly\ g(0) = 0.$

$Then\ g'(x) = 2f(x)\int_0^x f(t)dt - [f(x)]^3 = f(x)\left[2\int_0^x f(t)dt - \{f(x)\}^2\right].$

$Since\ f(0) = 0\ and\ f'(x) \in [0,1], f(x) \geq 0\ whenever\ x \geq 0\ (by\ lemma).$

$Let\ h(x) = 2\int_0^x f(t)dt - \{f(x)\}^2.\ Clearly\ h(0) = 0.$

$h'(x) = 2f(x) - 2f(x)f'(x) = 2f(x)[1 - f'(x)] \geq 0\ whenever\ x \geq 0\ \because f'(x) \in [0,1].$

$Since\ h(0) = 0\ and\ h'(x) \geq 0\ whenever\ x \geq 0, h(x) \geq 0\ whenever\ x \geq 0\ (by\ lemma).$

$g(0) = 0\ and\ g'(x) = f(x)h(x) \geq 0\ whenever\ x \geq 0\ \Rightarrow g(x) \geq 0\ for\ x \geq 0.$

$\therefore g(1) = \left[\int_0^1 f(x)dx\right]^2 - \int_0^1 [f(x)]^3 dx \geq 0, i.e. \left[\int_0^1 f(x)dx\right]^2 \geq \int_0^1 [f(x)]^3 dx.$