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A sum of the number of factorizations

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1 Formulation of Unordered Factorizations

For $n \geq 2$, let X_n be the set of unordered factorizations of n into divisors larger than 1. The size of X_n is equal to $a(n)$ by definition. Observe that an unordered factorization can be interpreted as data of a sequence of exponents $e_d \in \mathbb{Z}_{\geq 0}$ for $d \geq 2$ and $d \mid n$ such that $n = \prod_{d|n} d^{e_d}$. This implies that an exponent sequence uniquely determines an unordered factorization, and vice versa.

Let E be the set of all sequences of nonnegative integers $e = \{e_d\}_{d=2}^{\infty}$ starting at index 2 such that there exists some $N = N(e) \geq 1$ that makes $e_n = 0$ for every $n > N$. Define a function $\pi : E \to \mathbb{Z}_{>0}$ by $e \mapsto \prod_{d=2}^{\infty} d^{e_d} = \prod_{d=2}^{N(e)} d^{e_d}$. Because every element of E is finitely supported, the product is finite and this map is well defined.

Observe that there is a bijection $X_n \cong \pi^{-1}(n)$ for $n \geq 2$, as we discussed about formulation of unordered factorization. As convention, let $a(1) := 1$, in order to make $X_1 \cong \pi^{-1}(1)$ where $\pi^{-1}(1)$ is the singleton set of the sequence of zeros. $a(1) = 1$ can be justified by considering 1 to have one empty factorization. This does not matter to the answer as the desired sum $\sum_{n=2}^{\infty} \frac{a(n)}{n^2}$ starts from $n = 2$. Also, let identify X_n as the subset $\pi^{-1}(n)$ of E using this bijection.

2 Intuitive Result with Naïve Infinite Sum

To get a result not rigorous but intuitive, let construct an equation containing the desired sum:

$$1 + \sum_{n=2}^{\infty} \frac{a(n)}{n^2} = \sum_{n=1}^{\infty} \frac{a(n)}{n^2} = \sum_{n=1}^{\infty} \sum_{e \in X_n} \frac{1}{n^2} = \sum_{n=1}^{\infty} \sum_{e \in X_n} \frac{1}{\pi(e)^2}.$$

The last sum is taken over E; note that E is countable, as it is a countable union of finite sets X_n. If the sum $\sum_{e \in E} \frac{1}{\pi(e)^2}$ is convergent, then it converges absolutely and can be reordered in arbitrary way. For $N \in \mathbb{Z}_{\geq 1}$, let

$$E_N = \{e \in E : \forall n.(n > N \implies e_n = 0)\} \subset E$$

be the set of exponent sequences that vanishes after N. As $E_N \to E$ as $N \to \infty$, one has $\sum_{e \in E_N} \frac{1}{\pi(e)^2} \to \sum_{e \in E} \frac{1}{\pi(e)^2}$. Observe that the sum $\sum_{e \in E_N} \frac{1}{\pi(e)^2}$ is equal to $\frac{2N}{N+1}$. For $N = 1$, $\sum_{e \in E_1} \frac{1}{\pi(e)^2} = 1 = \frac{2 \cdot 1}{1+1}$. For $N \geq 2$, with the induction hypothesis $\sum_{e \in E_{N-1}} \frac{1}{\pi(e)^2} = \frac{2(N-1)}{N}$, the sum is

$$\begin{aligned}
\sum_{e \in E_N} \frac{1}{\pi(e)^2} &= \sum_{e_2=0}^{\infty} \sum_{e_3=0}^{\infty} \cdots \sum_{e_N=0}^{\infty} \frac{1}{\pi(e)^2} \\
&= \sum_{e_2=0}^{\infty} \sum_{e_3=0}^{\infty} \cdots \sum_{e_N=0}^{\infty} \frac{1}{\prod_{d=2}^{N} d^{2e_d}} \\
&= \sum_{e_2=0}^{\infty} \sum_{e_3=0}^{\infty} \cdots \sum_{e_{N-1}=0}^{\infty} \frac{1}{\prod_{d=2}^{N-1} d^{2e_d}} \sum_{e_N=0}^{\infty} \cdot \left(\frac{1}{N^{2e_N}} \right) \\
&= \sum_{e \in E_{N-1}} \frac{1}{\pi(e)^2} \cdot \frac{N^2}{N^2-1} \\
&= 2 \cdot \frac{N-1}{N} \cdot \frac{N^2}{(N-1)(N+1)} = \frac{2N}{N+1}.
\end{aligned}$$

Then $\sum_{e \in E} \frac{1}{\pi(e)^2} = \lim_{N \to \infty} \sum_{e \in E_N} \frac{1}{\pi(e)^2} = \lim_{N \to \infty} \frac{2N}{N+1} = 2$, hence $\sum_{n=2}^{\infty} \frac{a(n)}{n^2} = -1 + 2 = 1$ is deduced.

This solution is problematic as the sum over an infinte set is used without rigorous justification. For example, in the undergraduate level of analysis, one can sum only sequences of real or complex numbers. Hence the sum $\sum_{x \in S} f(x)$ over a countable set S must be followed by some explicit bijective "sequencification" $q : S \to \mathbb{Z}_{\geq 1}$ and be taken as $\sum_{n=1}^{\infty} f(q^{-1}(n))$, while the above solution does not give such.

3 Proof

There is an alternative approach that uses only infinite series. Let recall the concepts of convergence and infinite series from analysis, and some theorems without proof.

Definition 3.1 (Convergence of sequence)**.** Let $\{a_n\}$ be a sequence of real numbers. It is said to *converges to* L if, for arbitrary $\varepsilon > 0$, there exists $N \in \mathbb{Z}_{\geq 1}$ such that $|a_n - L| < \varepsilon$ for every $n > N$. This is denoted by $\lim_{n \to \infty} a_n = L$. $\{a_n\}$ *converges* if there exists some L such that it is converge to L.

Definition 3.2 (Infinite series)**.** Let $\{a_n\}$ be a sequence of real numbers. If the sequence of partial sums $\{\sum_{i=1}^{n} a_i\}$ converges to L, it is said that the infinite series $\sum_{n=1}^{\infty} a_n$ *converges to* L, and denoted by $\sum_{n=1}^{\infty} a_n = L$. $\sum_{n=1}^{\infty} a_n$ *converges* if there exists some L such that $\sum_{n=1}^{\infty} a_n$ converges to L.

Theorem 3.3 (Monotone convergence theorem)**.** *Suppose $\{a_n\}$ is an increasing sequence, i.e. $a_n \le a_m$ for any $n \le m$. If there is a real number M such that $a_n < M$ for every n, then a_n converges and $\lim\limits_{n\to\infty} a_n \le M$.*

Corollary 3.4. *Let $\{a_n\}$ be a sequence of positive real numbers. If every nth partial sum is bounded by some $M > 0$, i.e. $\sum_{i=1}^{n} a_i \le M$ for every n, then the infinite series $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n \le M$.*

Theorem 3.5. *Let $\{a_n\}, \{b_n\}$ be convergent sequences with limits A, B, respectively. Then $\{a_n b_n\}$ converges to AB.*

Firstly, let investigate the upper bound for $\sum_{n=1}^{\infty} \frac{a(n)}{n^2}$. For $N \in \mathbb{Z}_{\ge 1}$ and $M \in \mathbb{Z}_{\ge 0}$, let

$$E_{N,M} = \{e \in E : \forall n.(e_n \le M) \wedge (n > N \implies e_n = 0)\} \subset E_N$$

be the set of exponent sequences such that vanishes after N, and every component is bounded by M. As $E_{N,M}$ is a finite set of $(M+1)^{N-1}$ elements, the sum over $E_{N,M}$ is well defined. Observe that $E_{N,N}$ contains X_n for $1 \le n \le N$. For $e \in X_n$ and $2 \le d \le N$, one has

$$e_d \le d^{e_d} \le \prod_{d=2}^{N} d^{e_d} = \pi(e) = n \le N$$

hence $e \in E_{N,N}$. The partial sum $\sum_{n=1}^{N} \frac{a(n)}{n^2}$ is bounded by $\sum_{e \in E_{N,N}} \frac{1}{\pi(e)^2}$:

$$\sum_{n=1}^{N} \frac{a(n)}{n^2} = \sum_{n=1}^{N} \sum_{e \in X_n} \frac{1}{\pi(e)^2} \le \sum_{e \in E_{N,N}} \frac{1}{\pi(e)^2} = \sum_{e_2=0}^{N} \sum_{e_3=0}^{N} \cdots \sum_{e_N=0}^{N} \frac{1}{\pi(e)^2}.$$

From the definition $\pi(e) = \prod_{d=2}^{N} d^{e_d}$, one has

$$\sum_{e \in E_{N,N}} \frac{1}{\pi(e)^2} = \sum_{e_2=0}^{N} \sum_{e_3=0}^{N} \cdots \sum_{e_N=0}^{N} \frac{1}{\prod_{d=2}^{N} d^{2e_d}} = \prod_{d=2}^{N} \left(\sum_{e_d=0}^{N} \frac{1}{d^{2e_d}} \right).$$

This sum is bounded by *the sum over E_N*:

$$\prod_{d=2}^{N} \left(\sum_{e_d=0}^{N} \frac{1}{d^{2e_d}} \right) \le \prod_{d=2}^{N} \left(\sum_{e_d=0}^{\infty} \frac{1}{d^{2e_d}} \right) = \prod_{d=2}^{N} \frac{d^2}{d^2-1} = \frac{2N}{N+1}.$$

As $\frac{2N}{N+1} \le 2$, deduce that $\sum_{n=1}^{\infty} \frac{a(n)}{n^2}$ converges and $\sum_{n=1}^{\infty} \frac{a(n)}{n^2} \le 2$ using corollary 3.4. Note that the limit $\lim\limits_{M\to\infty} \sum_{e \in E_{N,M}} \frac{1}{\pi(e)^2}$ is equal to $\frac{2N}{N+1}$, by theorem 3.5.

Claim that $\sum_{n=1}^{\infty} \frac{a(n)}{n^2}$ is equal to 2. Observe that $E_{N,M}$ is contained in $\coprod_{n=1}^{K} X_n$ where $K = (N!)^M$. For $e \in E_{N,M}$, one has

$$\pi(e) = \prod_{d=2}^{N} d^{e_d} \le \prod_{d=2}^{N} d^M = (N!)^M = K$$

hence $e \in \coprod_{n=1}^{K} X_n$. The sum over $E_{N,M}$ gives a lower bound of $\sum_{n=1}^{\infty} \frac{a(n)}{n^2}$:

$$\sum_{e \in E_{N,M}} \frac{1}{\pi(e)^2} \leq \sum_{n=1}^{K} \sum_{e \in X_n} \frac{1}{\pi(e)^2} = \sum_{n=1}^{K} \frac{a(n)}{n^2} \leq \sum_{n=1}^{\infty} \frac{a(n)}{n^2}.$$

Note that this bound holds for every N and M. Taking the limit $M \to \infty$, deduce that $\frac{2N}{N+1} \leq \sum_{n=1}^{\infty} \frac{a(n)}{n^2}$ for every N. Taking the limit $N \to \infty$, deduce that $\sum_{n=1}^{\infty} \frac{a(n)}{n^2} \geq 2$. Conclude that $\sum_{n=1}^{\infty} \frac{a(n)}{n^2} = 2$ and $\sum_{n=2}^{\infty} \frac{a(n)}{n^2} = 1$.