KAIST POW 2022-17
The smallest number of subsets

2020 김유준
kyujun02@kaist.ac.kr

Answer

We consider the meaning of the number we have to compute:

$$\sum_{i=1}^{n} \phi(n, i)$$

This is sum of least elements over all subsets of $[n] := \{1, 2, \cdots, n\}$.

For each $1 \leq j \leq n$, the number of subsets of $[n]$ whose smallest element is j is 2^{n-j}. This is because $n - j$ number of natural numbers $j+1, j+2, \cdots, n$ can be either in or not in the subset with least element k. Let $l(S)$ denote the least element of $S \subseteq [n]$. Then, $1 \leq l(S) \leq n$ for all $S \subseteq [n]$. By above arguments,

$$\begin{aligned}\sum_{i=1}^{n} \phi(n, i) &= \sum_{S \subseteq [n]} l(S) = \sum_{j=1}^{n} \sum_{\substack{S \subseteq [n] \\ l(S)=j}} l(S) \\ &= \sum_{j=1}^{n} \sum_{\substack{S \subseteq [n] \\ l(S)=j}} j = \sum_{j=1}^{n} j \times 2^{n-j}\end{aligned}$$

Let $a_n = \sum_{i=1}^{n} \phi(n, i) = \sum_{j=1}^{n} j \times 2^{n-j}$.

$$2a_n = 1 \times 2^n + 2 \times 2^{n-1} + 3 \times 2^{n-2} + \cdots + n \times 2^1 \quad (1)$$
$$a_n = 1 \times 2^{n-1} + 2 \times 2^{n-2} + \cdots + (n-1) \times 2^1 + n \times 2^0 \quad (2)$$

(1)-(2) gives $a_n = 2^n + 2^{n-1} + \cdots + 2^1 - n = 2^{n+1} - 2 - n$. Thus,

$$\sum_{i=1}^{n} \phi(n, i) = 2^{n+1} - 2 - n$$