POW 2022-16 Identity for continuous functions

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September 23, 2022

Exercise 1

For a positive integer n, find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\sum_{k=0}^{n} \binom{n}{k} f(x^{2^k}) = 0$$

for all $x \in \mathbb{R}$. Answer : $f(x) \equiv 0$

Proof. To begin with, consider $n = 1$; $f(x) + f(x^2) = 0$. Immediately we have $f(0) = f(1) = 0$ and $f(-x) = f(x)$. For $0 < x < 1, f(x) = -f(x^2) = f(x^4) = \cdots = (-1)^m f(x^{2^m}) \to 0$ by continuity. Similarly, $f(x) = -f(x^{1/2}) = f(x^{1/4}) = \cdots = (-1)^m f(x^{2^{-m}}) \to 0$ for all $x > 1$. Therefore $f(x) = 0$ for all $x \in \mathbb{R}$. For the purpose of mathematical induction, let us assume that the claim holds for a positive integer n. By the identity $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ for $1 \le k \le n$, we may express

$$\sum_{k=0}^{n+1} \binom{n+1}{k} f(x^{2^k}) = f(x) + \sum_{k=1}^{n} \binom{n}{k} f(x^{2^k}) + \sum_{k=1}^{n} \binom{n}{k-1} f(x^{2^k}) + f(x^{2^{n+1}}) = g(x) + g(x^2) = 0,$$

where $g(x) = \sum_{k=0}^{n} \binom{n}{k} f(x^{2^k})$. Then we have $g(x) \equiv 0$ and $f(x) \equiv 0$ by the induction hypothesis. Thus the claim also holds for $n + 1$ and by the mathematical induction the claim is true for all positive integers. $\square$