POW2022-15 A Determinant of Stirling Numbers of Second Kind

2022기영인

Throughout the paper, we are to denote the Stirling numbers of second kind by the notation $\left\{ {n \atop k} \right\}$.

Lemma. $\det\begin{bmatrix} 1^1 & 2^1 & \cdots & n^1 \\ 1^2 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1^n & 2^n & \cdots & n^n \end{bmatrix} = 1!\,2!\cdots n!\ \ \forall n \in \mathbb{N}.$

pf. We prove by mathematical induction.
i) $n = 1$: $\det[1] = 1!$
ii) assume that $\det\begin{bmatrix} 1^1 & 2^1 & \cdots & n^1 \\ 1^2 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1^n & 2^n & \cdots & n^n \end{bmatrix} = 1!\,2!\cdots n!$ for some $n \geq 1$.
Let $A_{n+1} = \begin{bmatrix} 1^1 & \cdots & (n+1)^1 \\ \vdots & \ddots & \vdots \\ 1^{n+1} & \cdots & (n+1)^{n+1} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_{n+1} \end{bmatrix}$.

$$\det A_{n+1}$$
$$= \det\begin{bmatrix} v_1 \\ v_2 - (n+1)v_1 \\ \vdots \\ v_{n+1} - (n+1)v_n \end{bmatrix} \because \text{ determinant function is (n+1)-linear and alternating}$$
$$= \det\begin{bmatrix} 1^1 & 2^1 & \cdots & n^1 & (n+1)^1 \\ -n1^1 & -(n-1)2^1 & \cdots & -1n^1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -n1^n & -(n-1)2^n & \cdots & -1n^n & 0 \end{bmatrix}$$
$$= (-1)^{(n+1)+1}(n+1)\det\begin{bmatrix} -n1^1 & -(n-1)2^1 & \cdots & -1n^1 \\ \vdots & \vdots & \ddots & \vdots \\ -n1^n & -(n-1)2^n & \cdots & -1n^n \end{bmatrix}$$
$$= (-1)^{2n+2}(n+1)!\det\begin{bmatrix} 1^1 & 2^1 & \cdots & n^1 \\ 1^2 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1^n & 2^n & \cdots & n^n \end{bmatrix}$$
$$= 1!\,2!\cdots(n+1)!$$

Lemma. $\sum_{i=1}^{k} P(k,i)\left\{ {n \atop i} \right\} = k^n$, where $P(k,i) = \frac{k!}{(k-i)!}$.

pf. Instead of a rigorous proof, we are to show this by using some combinatorial sense.
Consider we are placing n distinct objects in k distinct containers, allowing empty containers to exist. Now we count the number of cases by sorting them by the number of nonempty containers. Say there are i nonempty containers. The number of ways to choose such containers is $\binom{n}{i}$. Once the containers are chosen, the number of ways to distribute among them is $i!\left\{ {n \atop i} \right\}$, for $\left\{ {n \atop i} \right\}$ is computed without considering the type of containers. Thus, the number of ways to distribute n distinct objects into i nonempty containers is $P(k,i)\left\{ {n \atop i} \right\}$.

The preceding lemma directly gives the equation $\left\{ {n \atop k} \right\} + \sum_{i=1}^{k-1} \frac{1}{(k-i)!} \left\{ {n \atop i} \right\} = \frac{k^n}{k!}$.

Let $S = \begin{bmatrix} \left\{ {m+1 \atop 1} \right\} & \cdots & \left\{ {m+1 \atop n} \right\} \\ \vdots & \ddots & \vdots \\ \left\{ {m+n \atop 1} \right\} & \cdots & \left\{ {m+n \atop n} \right\} \end{bmatrix} = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n]$. i.e., $\alpha_j = \begin{bmatrix} \left\{ {m+1 \atop j} \right\} \\ \vdots \\ \left\{ {m+n \atop j} \right\} \end{bmatrix}$.

$$\begin{aligned}
&\det S \\
&= \det\left[\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n + \textstyle\sum_{i=1}^{n-1} \frac{1}{(n-i)!} \alpha_i\right] \because \text{ determinant function is n-linear and alternating} \\
&= \det\left[\alpha_1 \quad \cdots \quad \alpha_{n-1} + \textstyle\sum_{i=1}^{n-2} \frac{1}{(n-1-i)!} \alpha_i \quad \alpha_n + \sum_{i=1}^{n-1} \frac{1}{(n-i)!} \alpha_i\right] \\
&\qquad \vdots \\
&= \det\left[\alpha_1 \quad \alpha_2 + \textstyle\sum_{i=1}^{1} \frac{1}{(2-i)!} \alpha_i \quad \cdots \quad \alpha_n + \sum_{i=1}^{n-1} \frac{1}{(n-i)!} \alpha_i\right] \\
&= \det \begin{bmatrix} 1^{m+1}/1! & 2^{m+1}/2! & \cdots & n^{m+1}/n! \\ 1^{m+2}/1! & 2^{m+2}/2! & \cdots & n^{m+2}/n! \\ \vdots & \vdots & \ddots & \vdots \\ 1^{m+n}/1! & 2^{m+n}/2! & \cdots & n^{m+n}/n! \end{bmatrix} \because \left\{ {n \atop k} \right\} + \textstyle\sum_{i=1}^{k-1} \frac{1}{(k-i)!} \left\{ {n \atop i} \right\} = \frac{k^n}{k!} \\
&= \frac{1^{\mathrm{m}} 2^m \cdots n^m}{1!\,2! \cdots n!} \det \begin{bmatrix} 1^1 & 2^1 & \cdots & n^1 \\ 1^2 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1^n & 2^n & \cdots & n^n \end{bmatrix} \\
&= \frac{(n!)^m}{1!\,2! \cdots n!} \det \begin{bmatrix} 1^1 & 2^1 & \cdots & n^1 \\ 1^2 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1^n & 2^n & \cdots & n^n \end{bmatrix} \\
&= (n!)^m \because lemma
\end{aligned}$$