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The number of eigenvalues of a symmetric matrix

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Claim that A has n positive eigenvalues with multiplicity.

Assume that $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \in \mathbb{R}^{2n}$ is an eigenvector of $A = \begin{bmatrix} O & B \\ B^T & C \end{bmatrix} \in M_{2n}(\mathbb{R})$ where $\mathbf{v}_i$ is n-dimensional vectors for $i \in \{1, 2\}$. Let λ be the eigenvalue of $\mathbf{v}$. Observe that

$$A\mathbf{v} = \begin{bmatrix} O & B \\ B^T & C \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} B\mathbf{v}_2 \\ B^T\mathbf{v}_1 + C\mathbf{v}_2 \end{bmatrix} = \lambda\mathbf{v} = \begin{bmatrix} \lambda\mathbf{v}_1 \\ \lambda\mathbf{v}_2 \end{bmatrix}.$$

Note that the λ is always nonzero. Because B is invertible, one has $\mathbf{v}_2 = \lambda B^{-1}\mathbf{v}_1$. If nonzero $\mathbf{v}$ has the eigenvalue 0, then $\mathbf{v}_2 = \mathbf{0}$, $B^T\mathbf{v}_2 + C\mathbf{0} = \mathbf{0}$ and $\mathbf{v}_1 = (B^T)^{-1}\mathbf{0} = \mathbf{0}$ hence $\mathbf{v} = \mathbf{0}$, which contradicts.

As λ is nonzero, write above equation by $\mathbf{v}_1 = \frac{1}{\lambda}B\mathbf{v}_2$ and $\frac{1}{\lambda}B^TB\mathbf{v}_2 + C\mathbf{v}_2 = \lambda\mathbf{v}_2$. To find eigenvalues of A is equivalent to find λ such that $(\lambda^2 I - \lambda C - B^TB)\mathbf{v} = \mathbf{0}$ has a nontrivial solution, i.e. $\det(\lambda^2 I - \lambda C - B^TB) = 0$.

Note that B^TB and C are real symmetric matrix, hence so is A and eigenvalues of A are real.

Let note some basic facts about real symmetric matrices without proof.

Proposition 1. *Suppose that M is a real symmetric matrix. Then the eigenvalues of M are real, and M is diagonalizable by orthogonal matrix, i.e. there exists an orthogonal matrix U so that $M = U^T\Lambda U$ where Λ is a diagonal matrix.*

Proposition 2. *Let M be a real symmetric matrix. M has strictly positive eigenvalues if and only if M is positive definite.*

Observe that $M = B^TB$ is positive definite. For nonzero x, Bx is nonzero and $x^TB^TBx = |Bx|^2 > 0$. Choose a diagonalization $M = U^T\Lambda U$, and let $M^{-\frac{1}{2}} = U^T\Lambda^{-\frac{1}{2}}U$, where $\Lambda^{-\frac{1}{2}}$ is the diagonal matrix whose entries are the inverse square root of entries of Λ. Note that $\Lambda^{-\frac{1}{2}}$ and $M^{-\frac{1}{2}}$ are real symmetric matrix, as entries of Λ are strictly positive.

Observe that $M^{-\frac{1}{2}}CM^{-\frac{1}{2}}$ is real symmetric as $(M^{-\frac{1}{2}}CM^{-\frac{1}{2}})^T = M^{-\frac{1}{2}}C^TM^{-\frac{1}{2}} = M^{-\frac{1}{2}}CM^{-\frac{1}{2}}$. Choose orthogonal matrix V so that $V^TM^{-\frac{1}{2}}CM^{-\frac{1}{2}}V$ is diagonal.

Let $W = M^{-\frac{1}{2}}V$. One has $W^TMW = I$, and $W^TCW = D$ is a diagonal matrix.

$\det(\lambda^2 I - \lambda C - M) = \det(W^T(\lambda^2 I - \lambda C - M)W) = \det(\lambda^2 I - \lambda D - I) = \det(\operatorname{diag}(\lambda^2 - d_i\lambda - 1))$. The determinant is zero if and only if one of $\lambda^2 - d_i\lambda - 1 = 0$. By Vieta's theorem, there is n positive eigenvalues and n negative eigenvalues.

Consider what if we do not consider multiplicity. Let $1 \leq k \leq n$. Set B be the diagonal matrix with k distinct positive entries, and $C = 0$. One can easily check that A has k distinct positive eigenvalues.