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Inequality involving sums with different powers

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Observe that the series $\sum_{n=0}^{\infty} \frac{1}{(x+n)^2}$ and $\sum_{n=0}^{\infty} \frac{1}{(x+n)^3}$ are absolutely convergent on $x \geq 1$. Because of the absolute convergence, we can reorder the sum and rewrite the left hand side by

$$\sum_{n=0}^{\infty} \frac{1}{(x+n)^4} + 2\sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{(x+n)^2} \frac{1}{(x+m)^2}.$$

Rearrange the sum once more, we can write it by

$$2\sum_{n=0}^{\infty} \left(\left(\sum_{m=n}^{\infty} \frac{1}{(x+n)^2(x+m)^2} \right) - \frac{1}{2(x+n)^4} \right)$$

and

$$2\sum_{n=0}^{\infty} \frac{1}{(x+n)^2} \left(-\frac{1}{2(x+n)^2} + \sum_{m=n}^{\infty} \frac{1}{(x+m)^2} \right).$$

Proposition 1. *Let $x_0 \geq 1$ be a real number. Then the following inequality holds:*

$$\sum_{n=0}^{\infty} \frac{1}{(x_0+n)^2} \geq \frac{1}{2x_0^2} + \frac{1}{x_0}$$

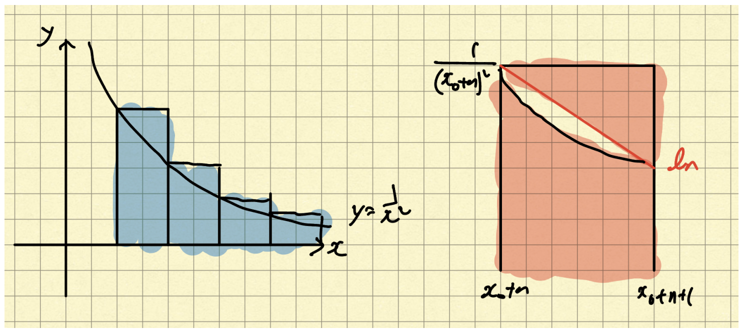
Proof. Let consider some subsets of $\mathbb{R}^2$ which is related to the graph of $y = \frac{1}{x^2}$. Let $A_n = (x_0 + n, x_0 + n + 1] \times (0, \frac{1}{(x_0+n)^2})$ for nonnegative integer n and $A = \bigcup_{n=0}^{\infty} A_n$. Note that the area of A is $\sum_{n=0}^{\infty} \frac{1}{(x_0+n)^2}$, the left hand side of the desired inequality.

Let $B = \{(x, y) : x > x_0 \wedge 0 < y < \frac{1}{x^2}\}$, $C_n = \{(x, y) : x_0 + n < x < x_0 + n + 1 \wedge l_n(x) < y < \frac{1}{(x_0+n)^2}\}$ where the graph of $y = l_n(x)$ is the unique line containing two points $(x_0 + n, \frac{1}{(x_0+n)^2})$ and $(x_0 + n + 1, \frac{1}{(x_0+n+1)^2})$. Since the function $y = \frac{1}{x^2}$ is convex, the line l_n is above the graph $y = \frac{1}{x^2}$ hence C_n is disjoint from B for every n. Also, note that each pair of C_n is disjoint. Deduce that the area of the union of B and C_n is the sum of area of B and C_n.

From the calculus, the area of B is equal to $\int_{x_0}^{\infty} \frac{1}{t^2} \mathrm{d}t = \frac{1}{x_0}$. The area of C_n is equal to $\frac{1}{2} \left(\frac{1}{(x_0+n)^2} - \frac{1}{(x_0+n+1)^2} \right)$. Summing all of that, the area of their union is $\frac{1}{2x_0^2} + \frac{1}{x_0}$.

Observe that B and C_n are contained in A, so their union is contained in A. We can deduce the desired inequality by comparing area of sets which are in the containing relation. $\square$

This result also can be shown (or at least seems feasible) by drawing the shape of A, B and C by hand.



The blue area on the left side represents A. The entire box on the right side represents A_n. There are two sections of red area: the above one represents C_n, and the below one represents $B \cap A_n$.

Using this result, the left hand side of the above formula has the lower bound

$$2\sum_{n=0}^{\infty} \frac{1}{(x+n)^2}\left(-\frac{1}{2(x+n)^2} + \left(\frac{1}{2(x+n)^2} + \frac{1}{(x+n)}\right)\right) = 2\sum_{n=0}^{\infty} \frac{1}{(x+n)^3}$$

which is obtained by replacing $\sum_{m=n}^{\infty} \frac{1}{(x+m)^2} = \sum_{m=0}^{\infty} \frac{1}{((x+n)+m)^2}$ into $\frac{1}{2(x+n)^2} + \frac{1}{(x+n)}$.