POW 2022-12 A partition of the power set of a set

Jo Yuri

Problem. *Consider the power set $P([n])$ consisting of 2^n subsets of $[n] = \{1, \cdots, n\}$. Find the smallest k such that the following holds: there exists a partition $Q_1, \cdots, Q_k$ of $P([n])$ so that there do not exist two distinct sets $A, B \in P([n])$ and $i \in [k]$ with $A, B, A \cup B, A \cap B \in Q_i$.*

Solution. The smallest such partition number is $k = n+1$. Suppose we have a partition $Q_1, \cdots, Q_k$ of $P([n])$. Then for any two distinct sets $A, B \in P([n])$ with $A \subset B$, there must exist $i \neq j \in [k]$ such that $A \in Q_i$ and $B \in Q_j$, i.e., A and B never belong to the same part. Otherwise, $A, B, A \cup B = A, A \cap B = B$ are contained in the same part, which is a contradiction. Now consider the chain $\emptyset \subset [1] \subset [2] \subset [3] \subset \cdots \subset [n]$ in $P([n])$. To split these $(n+1)$-many sets, at least $(n+1)$-many parts are needed. So $k \geq n+1$ and in fact $k = n+1$ is optimal if we take partition $\mathcal{P} = \{Q_0, \cdots, Q_n\}$ such that

$$Q_i := \{A \subset [n] : |A| = i\} \quad 0 \leq i \leq n.$$

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