POW 2022-11
Groups with torsions

2018 김기수

1 Brief Explanation

A countable group is embeddable in a 2-generated group. Consider the direct sum $C = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ where p runs over every prime. Observe that C has an element of order p for any prime p, and C is countable. Embedding it into some 2-generated group G. Then G is a finitely generated group which has a p-torsion element for any prime p. This is the desired result.

2 Solution

Definition 1. Let G be a group, A, B be isomorphic subgroups of G with an isomorphism $\phi : a \to b$. The *HNN extension* of G related to this data is the group with presentation

$$\langle G, t | t^{-1}at = \phi(a), \forall a \in A \rangle.$$

Theorem 1. *Any HNN extension of G embeds G.*

One may refer https://doi.org/10.1112/jlms/s1-24.4.247 for a proof.

Theorem 2. *Let C be a group with a countable underlying set. There exists a group G generated by two elements and an injective homomorphism $\iota : C \hookrightarrow G$.*

Also, one may refer https://doi.org/10.2307/2324618 for another proof.

Proof. Let $\{c_i : i \in \mathbb{Z}_{\geq 0}\}$ be the underlying set of C where c_0 is the identity element. Let $F = C * \langle a, b \rangle$ be the free product of C and the free group of generators $\{a, b\}$. Observe that the subgroups generated by $\{b^i ab^{-i} : i \in \mathbb{Z}_{\geq 0}\}$ and $\{c_i a^i ba^{-i} : i \in \mathbb{Z}_{\geq 0}\}$ are free groups with countable generating sets, hence they are isomorphic. Consider the HNN extension of F with these isomorphic subgroups:

$$\langle F, t | t^{-1} b^i ab^{-i} t = c_i a^i ba^{-i} \forall i \in \mathbb{Z}_{\geq 0} \rangle.$$

This embeds F hence it also embeds C. Since $b = tat^{-1}$ and $c_i = t^{-1} b^i ab^{-i} ta^{-i} b^{-1} a^i$, every element of F is generated by t and a. Therefore, this group is generated by two elements. □

Let $C = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ be the direct sum of cyclic groups of order p, where p runs over every prime number. Observe that the underlying set of C is the union of the increasing sequence

$$S_n = \{(x_p) \in C : x_p = 0 \text{ for } p > n\}.$$

Each S_n is finite hence the union is countable. By the theorem, there exists a group G generated by two elements, in particular which is finitely generated, and an injective homomorphism $\iota : \bigoplus_p \mathbb{Z}/p\mathbb{Z} \hookrightarrow G$. Let p be a prime. Observe that $x = (x_p) \in C$ given by $x_p = 1$ and $x_q = 0$ for $q \neq p$ is the element of order p. Since ι is injective, $\iota(x)$ is not trivial. As $[p]x = 0$, $[p]\iota(x) = \iota([p]x) = \iota(0) = 0$ and $\iota(x)$ is an element of G which has order p.