POW 1a

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Answer: Yes.

pf. Consider n linear forms.

(Similar to proof of lemma 5 in reference 1)

$$y_1 = x_0 + x_1 + \cdots + x_{n^2}$$

$$y_2 = 1x_1 + 2x_2 + \cdots + n^2 x_{n^2}$$

$$\vdots$$

$$y_n = \binom{0}{n-1} x_1 + \binom{1}{n-1} x_2 + \cdots + \binom{n^2}{n-1} x_{n^2}$$

$(x_i \in \mathbb{Z},\ 0 \le x_i \le n^2$ for $i = 1 \sim n)$.

Then, $0 \le y_1 \le n^2(1 + \cdots + 1) = n^2\binom{n^2+1}{1}$

$$0 \le y_2 \le n^2(1 + \cdots + n^2) = n^2\binom{n^2+1}{2}$$

$$\vdots$$

$$0 \le y_n \le n^2\left(\binom{n-1}{n-1} + \cdots + \binom{n^2}{n-1}\right) = n^2\binom{n^2+1}{n}$$

and $y_1 \sim y_n \in \mathbb{Z}$ ($\because \forall x_i,\ \binom{n}{r} \in \mathbb{Z}$)

$$\prod_{k=1}^{n} (\# \text{ of } y_k \text{ value}) = \prod_{k=1}^{n} \left(n^2 \binom{n^2+1}{k} + 1\right)$$

$$\le \prod_{k=1}^{n} n^2 \left(\binom{n^2+1}{k} + 1\right)$$

$$\le \prod_{k=1}^{n} n^2 \cdot n^{2k} \quad (\text{if } n \ge 2)$$

$$= n^{n^2+3n}$$

$$\# \text{ of } (x_0, \dots, x_{n^2}) \text{ pairs} = (n^2+1)^{n^2+1}$$

$$\ge n^{2(n^2+1)}$$

$$\ge n^{n^2+3n} \quad (\text{if } n \ge 2)$$

$$= \prod_{k=1}^{n} (\# \text{ of } y_k \text{ value})$$

Thus, # of $(y_1, y_2 \cdots y_n)$ pairs exceeds # of available value of $(y_1, y_2 \cdots y_n)$

$\therefore \exists (x_0' \cdots x_{n^2}') \ne (x_0'' \cdots x_{n^2}'')$ s.t. their $(y_1 \cdots y_n)$ values are same

Denote $z_0 = x_0'' - x_0'$, $\cdots$, $z_{n^2} = x_{n^2}'' - x_{n^2}'$.

$$\left.\begin{aligned} z_0 + z_1 + \cdots + z_{n^2} &= 0 \\ 1\cdot z_1 + \cdots + n^2 z_{n^2} &= 0 \\ &\vdots \\ \binom{0}{n-1} z_0 + \binom{1}{n-1} z_1 + \cdots + \binom{n^2}{n-1} z_{n^2} &= 0 \end{aligned}\right] \text{ (a)}$$

$(z_i \in \mathbb{Z}, -n^2 \le z_i \le n^2$ for $1 \le i \le n)$

Let $\phi(x) = z_0 + z_1 x + \cdots + z_{n^2} x^{n^2}$.

By (a), $\phi(1) = \phi^{(1)}(1) \cdots = \phi^{(n-1)}(1) = 0$.

Therefore, $\phi(x)$ is the polynomial that problem suggests.

$$\begin{aligned} \phi(1) &= z_0 + z_1 + \cdots + z_{n^2} = 0 \\ \phi^{(1)}(1) &= 1\cdot z_1 + \cdots + n^2 z_{n^2} = 0 \\ &\vdots \\ \phi^{(n-1)}(1) &= \binom{0}{n-1} z_0 + \binom{1}{n-1} z_1 + \cdots + \binom{n^2}{n-1} z_{n^2} = 0 \end{aligned}$$

Reference

1) On the roots of Certain Algebraic Equations
By A. BLOCH and G. POLYA.