09 Let $A_1, \ldots, A_k$ be presidential candidates in a country with $n \geq 1$ voters with $k \geq 2$. Candidates themselves are not voters. Each voter has her/his own preference on those $k$ candidates.

Find the maximum $m$ such that the following scenario is possible where $A_{k+1}$ indicates the candidate $A_1$: for each $i \in [k]$, there are at least $m$ voters who prefers $A_i$ to $A_{i+1}$.

**Solution.** Let $A = \{A_1, \ldots, A_k\}$ be the set of the $k$ candidates. Let $B = \{B_1, \ldots, B_n\}$ be the set of the $n$ voters. Denote $A_i < A_j$ if a voter prefers $A_i$ to $A_j$. Assume that every voter has her/his own preference such that $\prec$ is a strict total order on $A$. For example, it can be expressed as

$$B_1: A_2 \prec A_3 \prec \cdots \prec A_k \prec A_1$$

with $k - 1$ (immediate) comparisons. Hence, there are $n(k - 1)$ comparisons in total. For such scenario to be possible, $mk \leq n(k - 1)$. That is, $m \leq \lfloor n - n/k \rfloor$. (*)

(i) If $n \leq k$, then $\lfloor n - n/k \rfloor = n - 1$. The equality of (*) holds for the following scenario:

$$B_1: A_2 \prec A_3 \prec \cdots \prec A_k \prec A_1,$$

$$B_2: A_3 \prec A_4 \prec \cdots \prec A_k \prec A_1 \prec A_2,$$

$$\vdots$$

$$B_n: A_{n+1} \prec A_{n+2} \prec \cdots \prec A_k \prec A_1 \prec A_2 \prec \cdots \prec A_n.$$  

$n - 1$ voters—all but $B_i$—prefer $A_i$ to $A_{i+1}$ for each $i \in [n]$. All $n$ voters prefer $A_1$ to $A_{i+1}$ for each $i \in [k] \setminus [n]$.

(ii) If $n > k$, then there exist unique integers $q \geq 1$ and $r$ such that $n = kq + r$ and $0 \leq r < k$.

It follows from $n - q - 1 < n - \frac{n}{k} = n - q - \frac{r}{k} \leq n - q$ that

$$\lfloor n - n/k \rfloor = \begin{cases} n - q & \text{if } q \mid n, \\ n - q - 1 & \text{if } q \not\mid n. \end{cases}$$

The equality of (*) holds for the following scenario:

$$B_1, B_{k+1}, \ldots, B_{k(q-1)+1}, B_{kq+1}: A_2 \prec A_3 \prec \cdots \prec A_k \prec A_1,$$

$$B_2, B_{k+2}, \ldots, B_{k(q-1)+2}, B_{kq+2}: A_3 \prec A_4 \prec \cdots \prec A_k \prec A_1 \prec A_2,$$

$$\vdots$$

$$B_r, B_{k+r}, \ldots, B_{k(q-1)+r}, B_{kq+r}: A_{r+1} \prec A_{r+2} \prec \cdots \prec A_k \prec A_1 \prec A_2 \prec \cdots \prec A_r,$$

$$B_{r+1}, B_{k+r+1}, \ldots, B_{k(q-1)+r+1}: A_{r+2} \prec A_{r+3} \prec \cdots \prec A_k \prec A_1 \prec A_2 \prec \cdots \prec A_{r+1},$$

$$\vdots$$

$$B_k, B_{k+k}, \ldots, B_{k(q-1)+k}: A_1 \prec A_2 \prec \cdots \prec A_k.$$
$n - q - 1$ voters—all but $B_j$ with $j \equiv i \pmod{k}$—prefer $A_i$ to $A_{i+1}$ for each $i \in [r]$. (Note that $[0] = \emptyset$.) $n - q$ voters—all but $B_j$ with $j \equiv i \pmod{k}$—prefer $A_i$ to $A_{i+1}$ for each $i \in [k] \setminus [r]$.

Therefore, the maximum $m$ is $\lfloor n - n/k \rfloor$. \qedsymbol