09 Let $A_1, \cdots, A_k$ be presidential candidates in a country with $n \geq 1$ voters with $k \geq 2$. Candidates themselves are not voters. Each voter has her/his own preference on those k candidates.

Find the maximum m such that the following scenario is possible where A_{k+1} indicates the candidate A_1: for each $i \in [k]$, there are at least m voters who prefers A_i to A_{i+1}.

Solution. Let $\mathcal{A} = \{A_1, \cdots, A_k\}$ be the set of the k candidates. Let $\mathcal{B} = \{B_1, \cdots, B_n\}$ be the set of the n voters. Denote $A_i \prec A_j$ if a voter prefers A_i to A_j. Assume that every voter has her/his own preference such that $\prec$ is a strict total order on $\mathcal{A}$. For example, it can be expressed as

$$B_1: \quad A_2 \prec A_3 \prec \cdots \prec A_k \prec A_1$$

with $k-1$ (immediate) comparisons. Hence, there are $n(k-1)$ comparisons in total. For such scenario to be possible, $mk \leq n(k-1)$. That is, $m \leq \lfloor n - n/k \rfloor$. $(*)$

(i) If $n \leq k$, then $\lfloor n - n/k \rfloor = n - 1$. The equality of $(*)$ holds for the following scenario:

$$\begin{aligned}
B_1 &: \quad A_2 \prec A_3 \prec \cdots \prec A_k \prec A_1, \\
B_2 &: \quad A_3 \prec A_4 \prec \cdots \prec A_k \prec A_1 \prec A_2, \\
&\vdots \\
B_n &: \quad A_{n+1} \prec A_{n+2} \prec \cdots \prec A_k \prec A_1 \prec A_2 \prec \cdots \prec A_n.
\end{aligned}$$

$n-1$ voters—all but B_i—prefer A_i to A_{i+1} for each $i \in [n]$. All n voters prefer A_i to A_{i+1} for each $i \in [k] \setminus [n]$.

(ii) If $n > k$, then there exist unique integers $q \geq 1$ and r such that $n = kq + r$ and $0 \leq r < k$. It follows from

$$n - q - 1 < n - \frac{n}{k} = n - q - \frac{r}{k} \leq n - q$$

that

$$\lfloor n - n/k \rfloor = \begin{cases} n - q & \text{if } q \mid n, \\ n - q - 1 & \text{if } q \nmid n. \end{cases}$$

The equality of $(*)$ holds for the following scenario:

$$\begin{aligned}
B_1, B_{k+1}, \cdots, B_{k(q-1)+1}, B_{kq+1} &: \quad A_2 \prec A_3 \prec \cdots \prec A_k \prec A_1, \\
B_2, B_{k+2}, \cdots, B_{k(q-1)+2}, B_{kq+2} &: \quad A_3 \prec A_4 \prec \cdots \prec A_k \prec A_1 \prec A_2, \\
&\vdots \\
B_r, B_{k+r}, \cdots, B_{k(q-1)+r}, B_{kq+r} &: \quad A_{r+1} \prec A_{r+2} \prec \cdots \prec A_k \prec A_1 \prec A_2 \prec \cdots \prec A_r, \\
B_{r+1}, B_{k+r+1}, \cdots, B_{k(q-1)+r+1} &: \quad A_{r+2} \prec A_{r+3} \prec \cdots \prec A_k \prec A_1 \prec A_2 \prec \cdots \prec A_{r+1}, \\
&\vdots \\
B_k, B_{k+k}, \cdots, B_{k(q-1)+k} &: \quad A_1 \prec A_2 \prec \cdots \prec A_k
\end{aligned}$$

$n-q-1$ voters—all but B_j with $j \equiv i \pmod{k}$—prefer A_i to A_{i+1} for each $i \in [r]$. (Note that $[0] = \varnothing$.) $n-q$ voters—all but B_j with $j \equiv i \pmod{k}$—prefer A_i to A_{i+1} for each $i \in [k] \setminus [r]$.

Therefore, the maximum m is $\lfloor n - n/k \rfloor$. □