POW 2022-08
Two sequences

2018 김기수

At $n = 80143857 > 3$, one has $a_n = 25510582$ and $b_n = 25510583$ so that $a_n \neq b_n$.

Proposition 1 (approximation of π)**.** $3.14159265358979323846 < \pi < 3.14159265358979323847$.

See https://oeis.org/A000796.

Observe that $\frac{80143857}{\pi} < \frac{10^{20} \cdot 80143857}{314159265358979323846} < 25510582$. The first inequality is from the previous proposition. The second inequality is derived from the direct calculation:

$$80143857 \cdot 10^{20} < 8014385700000001477277938372 = 25510582 \cdot 314159265358979323846.$$

Lemma 1. *Let $g : [0,1] \to \mathbb{R}$ be a differentiable function such that $g(0) = 0$ and $g'(x) < 0$ for $0 < x < 1$. Then $g(x) \leq 0$ in $(0,1)$.*

This is the direct result of the fundamental theorem of calculus.

Proposition 2. *For $0 < x < 1$, $\sin x \leq x - \frac{1}{6}x^3 + \frac{1}{120}x^5$.*

Proof. Let investigate the funciton $g(x) = \sin x - x + \frac{1}{6}x^3 - \frac{1}{120}x^5$. One has $g^{(5)}(x) = \cos x - 1 < 0$ on $0 < x < 1$. Note that $g^{(n)}(0) = 0$ for $n \in \{0, 1, 2, 3, 4\}$. Repeatedly applying the previous lemma, one obtains the desired result. □

Let $x = \frac{314159265358979323847}{10^{20} \cdot 80143857}$. Obviously $0 < x < 1$. One has $\sin \frac{\pi}{80143857} < \sin x \leq \frac{1}{x - \frac{1}{6}x^3 + \frac{1}{120}x^5}$ and $\frac{1}{x - \frac{1}{6}x^3 + \frac{1}{120}x^5} < \frac{1}{\sin \frac{\pi}{80143857}}$ from the previous proposition. The direct calculation shows that $25510582 < \frac{1}{x - \frac{1}{6}x^3 + \frac{1}{120}x^5}$ hence $25510582 < \frac{1}{\sin \frac{\pi}{80143857}}$.

Therefore, at $n = 80143857$, one has

$$\frac{n}{\pi} < 25510582 < \frac{1}{\sin \frac{\pi}{n}}$$

so that their ceilings are different.