POW 2022-07 Coulomb potential

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Problem. *Prove the following identity for $x, y \in \mathbb{R}^3$:*

$$\frac{1}{|x-y|} = \frac{1}{\pi^3}\int_{\mathbb{R}^3} \frac{1}{|x-z|^2}\frac{1}{|y-z|^2}dz.$$

Solution. By substituting $z = x + z'$, we see that $\int_{\mathbb{R}^3} \frac{1}{|x-z|^2}\frac{1}{|y-z|^2}dz = \int_{\mathbb{R}^3} \frac{1}{|z|^2}\frac{1}{|y-x-z|^2}dz.$ Hence it is sufficient to consider for $x \neq 0$

$$F(x) := \frac{1}{\pi^3}\int_{\mathbb{R}^3} \frac{1}{|z|^2}\frac{1}{|x-z|^2}dz.$$

Before we begin our proof, let us observe that the integration $F(x)$ is well defined for $x \neq 0$; two singularities $z = 0$ and $z = x$ can be handeld as

- If $|z| < |x|/2$, then $|x - z| > |x|/2$ thus $\int_{|z|<|x|/2} \frac{1}{|z|^2}\frac{1}{|x-z|^2}dz \le C/|x|.$
- If $|x - z| < |x|/2$, then $|z| > |x|/2$ thus $\int_{|x-z|<|x|/2} \frac{1}{|z|^2}\frac{1}{|x-z|^2}dz \le C/|x|.$

Finally, if $|z| > 2|x|$, then $|x - z| \ge |z| - |x| > |z|/2$ thus

$$\int_{|z|>2|x|} \frac{1}{|z|^2}\frac{1}{|x-z|^2}dz \le C/|x|.$$

In addition, $F(x)$ is spherically symmetric, i.e., there exists a function $f : (0, \infty) \to \mathbb{R}$ such that $F(x) = f(|x|)$ for all $x \neq 0$. The proof is easy; for any $U \in O(3)$, by substituting $z = Uz'$,

$$F(Ux) = \frac{1}{\pi}\int_{\mathbb{R}^3} \frac{1}{|z|^2}\frac{1}{|Ux-z|^2}dz = \frac{1}{\pi}\int_{\mathbb{R}^3} \frac{1}{|Uz'|^2}\frac{1}{|Ux-Uz'|^2}|\det U|dz' = F(x).$$

It allows us to restrict $x = (r, 0, 0)$ for simplicity. Then by Fubini's theorem and polar transform,

$$\begin{aligned} f(r) &= \frac{1}{\pi^3}\int_{\mathbb{R}^3} \frac{1}{z_1^2+z_2^2+z_3^2}\cdot\frac{1}{(z_1-r)^2+z_2^2+z_3^2}dz_1dz_2dz_3 \\ &= \frac{1}{\pi^3}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2} \frac{1}{z_1^2+z_2^2+z_3^2}\cdot\frac{1}{(z_1-r)^2+z_2^2+z_3^2}dz_2dz_3\right)dz_1 \\ &= \frac{1}{\pi^3}\int_{\mathbb{R}}\left(2\pi\int_0^\infty \frac{1}{z_1^2+\rho^2}\cdot\frac{1}{(z_1-r)^2+\rho^2}\cdot\rho d\rho\right)dz_1, \quad r > 0. \end{aligned}$$

The following fact is useful ;

$$\int_0^\infty \frac{1}{a^2+\rho^2}\cdot\frac{1}{b^2+\rho^2}\cdot\rho d\rho = \int_0^\infty \frac{1}{b^2-a^2}\left(\frac{\rho}{a^2+\rho^2}-\frac{\rho}{b^2+\rho^2}\right)d\rho = \frac{1}{2}\cdot\frac{\log b^2-\log a^2}{b^2-a^2}.$$

Hence

$$f(r) = \frac{1}{\pi^2}\int_{-\infty}^\infty \frac{\log|z_1-r|^2-\log|z_1|^2}{|z_1-r|^2-|z_1|^2}dz_1.$$

Let us prove $f(r) = 1/r$.

By substituting $z_1 = rw$,

$$f(r) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{\log|rw-r|^2 - \log|rw|^2}{|rw-r|^2 - |rw|^2} r\,dw = \frac{1}{r}\cdot\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{\log|w-1|^2-\log|w|^2}{|w-1|^2-|w|^2}dw.$$

If we can show $f(1) = 1$, the proof is completed. By substituting $w = \frac{1+x}{2}$,

$$\begin{aligned} f(1) &= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{\log|w-1|^2-\log|w|^2}{|w-1|^2-|w|^2}dw = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{\log|1+x|-\log|1-x|}{x}dx \\ &= \frac{2}{\pi^2}\left(\int_0^1 \frac{\log|1+x|-\log|1-x|}{x}dx + \int_1^{\infty}\frac{\log|1+x|-\log|1-x|}{x}dx\right). \end{aligned}$$

Note that the above two integrals are the same ; set $x = 1/y$

$$\int_0^1 \frac{\log|1+x|-\log|1-x|}{x}dx = \int_1^{\infty}\frac{\log|1+y|-\log|1-y|}{y}dy.$$

Hence by putting $x = \frac{y+1}{y-1}$,

$$f(1) = \frac{4}{\pi^2}\int_1^{\infty}\frac{\log|1+x|-\log|1-x|}{x}dx = \frac{8}{\pi^2}\int_1^{\infty}\frac{\log y}{y^2-1}dy = \frac{4}{\pi^2}\int_0^{\infty}\frac{\log y}{y^2-1}dy = 1,$$

from the following **Lemma** and simple fact $\int_0^1 \frac{\log y}{y^2-1}dy = \int_1^{\infty}\frac{\log x}{x^2-1}dy$. ($y = 1/x$) It ends the proof. □

Lemma. $\int_0^{\infty}\frac{\log y}{y^2-1}dy = \frac{\pi^2}{4}$.

Proof. We evaluate the integral by the method of complex analysis. Taking the usual branch cut along the negative real-axis ($\mathrm{Arg}\, z \in (-\pi,\pi)$), define $\phi(z) = \frac{\log z}{z^2-1}$. We integrate $\phi(z)$ over the counterclockwise contour $C = C_1 \cup C_2 \cup C_3 \cup C_4$ given by ($\epsilon \to 0^+, R \to +\infty$),

- $C_1 : z = t;\;\; \epsilon \le t \le R$,
- $C_2 : z = Re^{i\theta};\;\; 0 \le \theta \le \pi/2$,
- $C_3 : z = it;\;\; \epsilon \le t \le R$,
- $C_4 : z = \epsilon e^{i\theta};\;\; 0 \le \theta \le \pi/2$.

Since ϕ is analytic in the region enclosed by C (note that $z = 1$ is a removable singularity), **Cauchy Integral Theorem** is applied ; $\int_C \phi(z)dz = \sum_{k=1}^{4}\int_{C_k}\phi(z)dz = 0$.

Now we are taking the limits $\epsilon \to 0^+, R \to +\infty$ so that

1. $\left|\int_{C_2}\phi(z)dz\right| \le \frac{\pi}{2}R\cdot\frac{\log R + \pi/2}{R^2-1} \to 0$,
2. $\left|\int_{C_4}\phi(z)dz\right| \le \frac{\pi}{2}\epsilon\cdot\frac{|\log\epsilon| + \pi/2}{1-\epsilon^2} \to 0$,

On the other hand,

$$\int_{C_3}\phi(z)dz = \int_R^{\epsilon}\frac{\log(it)}{(it)^2-1}i\,dt = i\int_{\epsilon}^{R}\frac{\log(t)}{t^2+1}dt - \frac{\pi}{2}\int_{\epsilon}^{R}\frac{1}{t^2+1}dt \to -\frac{\pi^2}{4},$$

as $\int_0^{\infty}\frac{\log(t)}{t^2+1}dt = \int_0^1\frac{\log(t)}{t^2+1}dt + \int_1^{\infty}\frac{\log(t)}{t^2+1}dt = 0$ (to see this, set $t = 1/s$) and $\int_0^{\infty}\frac{1}{t^2+1}dt = \frac{\pi}{2}$. In conclusion,

$$\int_{C_1}\phi(z)dz \to \frac{\pi^2}{4}.$$

□