Problem. Prove the following identity for \( x, y \in \mathbb{R}^3 \):

\[
\frac{1}{|x-y|} = \frac{1}{\pi^3} \int_{\mathbb{R}^3} \frac{1}{|x-z|^2} \frac{1}{|y-z|^2} \, dz.
\]

Solution. By substituting \( z = x + z' \), we see that

\[
\int_{\mathbb{R}^3} \frac{1}{|x-z|^2} \frac{1}{|y-z|^2} \, dz = \int_{\mathbb{R}^3} \frac{1}{|z|^2} \frac{1}{|y-x|^2} \, dz.
\]

Hence it is sufficient to consider for \( x \neq 0 \)

\[
F(x) := \frac{1}{\pi^3} \int_{\mathbb{R}^3} \frac{1}{|z|^2} \frac{1}{|x-z|^2} \, dz.
\]

Before we begin our proof, let us observe that the integration \( F(x) \) is well defined for \( x \neq 0 \); two singularities \( z = 0 \) and \( z = x \) can be handled as

- If \( |z| < |x|/2 \), then \( |x-z| > |x|/2 \) thus
  \[
  \int_{|z|<|x|/2} \frac{1}{|z|^2} \frac{1}{|x-z|^2} \, dz \leq C/|x|.
  \]
- If \( |x-z| < |x|/2 \), then \( |z| > |x|/2 \) thus
  \[
  \int_{|x-z|<|x|/2} \frac{1}{|z|^2} \frac{1}{|x-z|^2} \, dz \leq C/|x|.
  \]

Finally, if \( |z| > 2|x| \), then \( |x-z| \geq |z|-|x| > |x|/2 \) thus

\[
\int_{|z|>2|x|} \frac{1}{|z|^2} \frac{1}{|x-z|^2} \, dz \leq C/|x|.
\]

In addition, \( F(x) \) is spherically symmetric, i.e., there exists a function \( f : (0, \infty) \to \mathbb{R} \) such that \( F(x) = f(|x|) \) for all \( x \neq 0 \). The proof is easy; for any \( U \in O(3) \), by substituting \( z = Uz' \),

\[
F(Ux) = \frac{1}{\pi^3} \int_{\mathbb{R}^3} \frac{1}{|z|^2} \frac{1}{|Ux-z'|^2} \, dz = \frac{1}{\pi^3} \int_{\mathbb{R}^3} \frac{1}{|Uz'|^2} \frac{1}{|Ux-Uz'|^2} \, dz.
\]

It allows us to restrict \( x = (r, 0, 0) \) for simplicity. Then by Fubini’s theorem and polar transform,

\[
f(r) = \frac{1}{\pi^3} \int_{\mathbb{R}^3} \frac{1}{z_1^2 + z_2^2 + z_3^2} \frac{1}{(z_1 - r)^2 + z_2^2 + z_3^2} \, dz_1 \, dz_2 \, dz_3
\]
\[
= \frac{1}{\pi^3} \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} \frac{1}{z_1^2 + z_2^2 + z_3^2} \frac{1}{(z_1 - r)^2 + z_2^2 + z_3^2} \, dz_2 \, dz_3 \right) \, dz_1
\]
\[
= \frac{1}{\pi^3} \int_{\mathbb{R}} 2\pi \int_0^\infty \frac{1}{z_1^2 + \rho^2} \frac{1}{(z_1 - r)^2 + \rho^2} \cdot \rho \, d\rho \, dz_1, \quad r > 0.
\]

The following fact is useful:

\[
\int_0^\infty \frac{1}{a^2 + \rho^2} \cdot \frac{1}{b^2 + \rho^2} \cdot \rho \, d\rho = \int_0^\infty \frac{1}{b^2 - a^2} \left( \frac{\rho}{a^2 + \rho^2} - \frac{\rho}{b^2 + \rho^2} \right) \, d\rho = \frac{1}{2} \cdot \frac{\log b^2 - \log a^2}{b^2 - a^2}.
\]

Hence

\[
f(r) = \frac{1}{\pi^3} \int_{-\infty}^\infty \frac{\log |z_1-r|^2 - \log |z_1|^2}{|z_1-r|^2 - |z_1|^2} \, dz_1.
\]

Let us prove \( f(r) = 1/r \).
By substituting $z = rw$, 
\[
    f(r) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\log |r - |r^2| - \log |rw|^2}{|r - |r^2| - |rw|^2} r dw = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\log |w - 1|^2 - \log |w|^2}{|w - 1|^2 - |w|^2} dw.
\]
If we can show $f(1) = 1$, the proof is completed. By substituting $w = \frac{1 + \pi i}{2}$, 
\[
    f(1) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\log |w - 1|^2 - \log |w|^2}{|w - 1|^2 - |w|^2} dw = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\log |1 + x| - \log |1 - x|}{x} dx
\]
\[
    = \frac{2}{\pi^2} \left( \int_{0}^{1} \frac{\log |1 + x| - \log |1 - x|}{x} dx + \int_{1}^{\infty} \frac{\log |1 + x| - \log |1 - x|}{x} dx \right).
\]
Note that the above two integrals are the same; set $x = 1/y$ 
\[
    \int_{0}^{1} \frac{\log |1 + x| - \log |1 - x|}{x} dx = \int_{1}^{\infty} \frac{\log |1 + y| - \log |1 - y|}{y} dy.
\]
Hence by putting $x = \frac{y + 1}{y - 1}$, 
\[
    f(1) = \frac{4}{\pi^2} \int_{1}^{\infty} \frac{\log |1 + x| - \log |1 - x|}{x} dx = \frac{8}{\pi^2} \int_{1}^{\infty} \frac{\log y}{y^2 - 1} dy = \frac{4}{\pi^2} \int_{0}^{\infty} \frac{\log y}{y^2 - 1} dy = 1,
\]
from the following Lemma and simple fact $\int_{0}^{1} \frac{\log y}{y^2 - 1} dy = \int_{1}^{\infty} \frac{\log x}{x^2 - 1} dx$. $(y = 1/x)$ It ends the proof. \[\square\]

**Lemma.** \[\int_{0}^{\infty} \frac{\log y}{y^2 - 1} dy = \frac{\pi^2}{4}.\]

**Proof.** We evaluate the integral by the method of complex analysis. Taking the usual branch cut along the negative real-axis $(\text{Arg } z \in (-\pi, \pi))$, define $\phi(z) = \frac{\log z}{z^2 - 1}$. We integrate $\phi(z)$ over the counterclockwise contour $C = C_1 \cup C_2 \cup C_3 \cup C_4$ given by $(\epsilon \to 0^+, R \to +\infty)$,

- $C_1 : z = t; \quad \epsilon \leq t \leq R$,
- $C_2 : z = Re^{i\theta}; \quad 0 \leq \theta \leq \pi/2$,
- $C_3 : z = it; \quad \epsilon \leq t \leq R$,
- $C_4 : z = \epsilon e^{i\theta}; \quad 0 \leq \theta \leq \pi/2$.

Since $\phi$ is analytic in the region enclosed by $C$ (note that $z = 1$ is a removable singularity), **Cauchy Integral Theorem** is applied: \[\int_{C} \phi(z) dz = \sum_{k=1}^{4} \int_{C_k} \phi(z) dz = 0.\]

Now we are taking the limits $\epsilon \to 0^+, R \to +\infty$ so that 

1. $\left| \int_{C_2} \phi(z) dz \right| \leq \frac{\pi}{2} R \cdot \frac{\log R + \pi/2}{R^2 - 1} \to 0$,
2. $\left| \int_{C_4} \phi(z) dz \right| \leq \frac{\pi}{2} \epsilon \cdot \frac{|\log \epsilon + \pi/2|}{1 - \epsilon^2} \to 0$.

On the other hand, 
\[
    \int_{C_3} \phi(z) dz = \int_{R}^{\epsilon} \log(it) \frac{1}{(it)^2 - 1} \ dt = \int_{\epsilon}^{R} \log(t) \frac{1}{t^2 + 1} dt - \frac{\pi}{2} \int_{\epsilon}^{R} \frac{1}{t^2 + 1} dt \to -\frac{\pi^2}{4},
\]
as \[\int_{0}^{\infty} \frac{\log(t)}{t^2 + 1} dt = \int_{0}^{1} \frac{\log(t)}{t^2 + 1} dt + \int_{1}^{\infty} \frac{\log(t)}{t^2 + 1} dt = 0 \text{ (to see this, set } t = 1/s) \text{ and } \int_{0}^{\infty} \frac{1}{t^2 + 1} dt = \frac{\pi}{2}.\]

In conclusion, 
\[\int_{C_1} \phi(z) dz \to \frac{\pi^2}{4}.\] \[\square\]