

* Show that there do not exist perfect squares a, b, c such that $a^2 + b^2 = c^2$, provided that a, b, c are nonzero integers.

pf) We introduce 2 lemmas for the proof.

Lemma 1. Let x, y, z be positive integers s.t. $\gcd(x, y, z) = 1$, $2|x$, $x^2 + y^2 = z^2$.

Then $(x, y, z) = (2st, s^2 - t^2, s^2 + t^2)$, where s, t are integers s.t. $s > t > 0$, $\gcd(s, t) = 1$ & $s \not\equiv t \pmod{2}$. We say such x, y, z a "Pythagorean triple", which is primitive.

$\because \gcd(x, y, z) = 1$)

pf of Lemma 1) \Leftrightarrow Suppose (x, y, z) be Pythagorean triple which is primitive.

Since x is even, y, z must be odd. (If y also even, $\cancel{z^2}$ be even also, which contradicts that $\gcd(x, y, z) = 1$.) Then, $z-y \equiv z+y \equiv 0 \pmod{2}$, so let $\begin{cases} z-y=2u \\ z+y=2v \end{cases}$

Then $x^2 = (z^2 - y^2) = (z-y)(z+y) \Rightarrow \underline{\left(\frac{x}{2}\right)^2 = uv}$. Notice that $\gcd(u, v) = 1$; suppose $\gcd(u, v) = d > 1$, then $d|u-v, d|u+v \Rightarrow d|\frac{z}{2}, d|y \Rightarrow$ contradiction to $\gcd(y, z) = 1$. From

\otimes , since $\gcd(u, v) = 1$, $u = s^2, v = t^2$ for some $s, t \in \mathbb{N}$. (Suppose $\frac{u}{v} > 1$, then

$u = p_1^{k_1} \dots p_r^{k_r}, v = q_1^{j_1} q_2^{j_2} \dots q_r^{j_r}$. Using $\gcd(u, v) = 1$, it is easy to find that $p_i \neq q_j$.

$\left(\begin{array}{l} p_i, q_j : \cancel{\text{are primes}} \\ p_i \neq p_j, q_i \neq q_j \end{array} \right) \quad \left(\begin{array}{l} \text{Since } uv = (\text{square number}), k_i \equiv 0 \pmod{2}, j_i \equiv 0 \pmod{2} \\ \Rightarrow \begin{cases} u = s^2 \\ v = t^2 \end{cases} \end{array} \right)$

Then, $z = u+v = s^2 + t^2$, $y = v-u = s^2 - t^2$, $x = 2st$. Indeed, as $\gcd(s, t)$ divides y & z ,

$\gcd(s, t) = 1$. If s, t are both even or odd, it leads to y, z both even; $s \not\equiv t \pmod{2}$.

$(s > t)$ follows from $\frac{t^2}{u} = \frac{z-y}{2} < \frac{z+y}{2} = \frac{s^2}{v}$) $\quad \text{ok}$

\Leftarrow Suppose $s, t \in \mathbb{N}$ described in the lemma. Then, $x^2 + y^2 = (2st)^2 + (s^2 - t^2)^2 = (s^2 + t^2)^2 = z^2$.

Assume $\gcd(x, y, z) = d > 1$. Observe that if p is some prime divisor of d ,

(Continued) $p \neq 2$, because p divides odd integer z ($s \not\equiv t \pmod{2}$) $\Rightarrow z \equiv 1 \pmod{2}$

From $p|y$ & $p|z$, $p|z+y$, $p|z-y \Rightarrow p|2s^2$, $p|2t^2$, since $p \neq 2$, $p|s^2$, $p|t^2 \Rightarrow p|s$, $p|t$
 $\Rightarrow \gcd(s, t) > 1$; contradiction. Hence, $d=1$ & (x, y, z) is a primitive Pythagorean triple.

Lemma 2. $x^4 + y^4 = z^2$ has no solution for positive integers $x, y, & z$.

Pf of Lemma 2) Suppose (x_0, y_0, z_0) satisfies $x_0^4 + y_0^4 = z_0^2$. Then, (x_0^2, y_0^2, z_0) satisfies $x^2 + y^2 = z^2$. WLOG. $\gcd(x_0, y_0) = 1$; otherwise, let $\gcd(x_0, y_0) = d$, $x_0 = dx_1$, $y_0 = dy_1$, $z_0 = d^2z_1$, so $x_1^4 + y_1^4 = z_1^2 \Rightarrow \gcd(x_1, y_1) = 1$. Now, as (x_0^2, y_0^2, z_0) satisfies the condition of Lemma 1, WLOG. x_0^2 (x_0) even, then $\exists s > t > 0$, $\gcd(s, t) = 1$ s.t. $x_0^2 = 2st$, $y_0^2 = s^2 - t^2$, $z_0 = s^2 + t^2$, where exactly one of $s+t$ is even. If s is even, $y_0^2 \equiv 3 \pmod{4}$, which is impossible; t must be even; let $t = 2r$. Then $x_0^2 = 4sr$, so $(\frac{x_0}{2})^2 = sr$. Then from the similar argument at the proof of Lemma 1, $s = z_1^2$, $r = w_1^2$ for some $z_1, w_1 \in \mathbb{N}$
 $\therefore \gcd(s, t) = 1 \Rightarrow \gcd(s, r) = 1$, as s :odd) Since $t^2 + y_0^2 = s^2$ & $\gcd(s, t) = 1$, $\gcd(y_0, t, s) = 1$, making t, y_0, s a primitive Pythagorean triple. As t even, $t = 2uv$, $y_0 = u^2 - v^2$, $s = u^2 + v^2$ $u, v \in \mathbb{N}$ s.t. $\gcd(u, v), u > v > 0$. Now $uv = \frac{t}{2} = r = w_1^2$ implies u, v both squares; let $u = x_1^2$, $v = y_1^2$. Then, $s = z_1^2 = u^2 + v^2 = x_1^4 + y_1^4$. As we assumed z_1 & t being positive, we have $0 < z_1 \leq z_1^2 = s \leq s^2 < s^2 + t^2 = z_0$. Starting with one solution x_0, y_0, z_0 of $x^4 + y^4 = z^2$, we've constructed another solution x_1, y_1, z_1 s.t. $0 < z_1 < z_0$. Repeating this argument, we would have a third solution x_2, y_2, z_2 s.t. $0 < z_2 < z_1 < z_0$. We can apply the argument infinitely, but we have only finite supply of positive integers less than z_0 . Hence, a contradiction.

Now, we prove the given statement. First, we may assume $\gcd(a,b,c)=1$.

If $\gcd(a,b,c)=d>1$, let $a=da_1$, $b=db_1$, $c=dc_1$, so $a^2+b^2=d^2a_1^2+d^2b_1^2=d^2c_1^2=c^2$
 $\Rightarrow a_1^2+b_1^2=c_1^2$ & $\gcd(a_1,b_1,c_1)=1$. Second, we may assume a,b,c are all positive
 integers. If (a,b,c) is a solution, $\{-a, \pm b, \pm c\}, (-a, \pm b, \mp c)$ are solutions of
 $\{a, \pm b, \pm c\}, (a, \pm b, \mp c)$

$a^2+b^2=c^2$. Now, we prove this by deriving a contradiction. Suppose

a_0, b_0, c_0 satisfies $a^2+b^2=c^2$ & a_0, b_0, c_0 are perfect squares & $\gcd(a_0, b_0, c_0)=1$,

$a_0, b_0, c_0 > 0$; let $a_0=x^2$, $b_0=y^2$, $c_0=z^2$, where $x, y, z \in \mathbb{N}$. Then, $x^4+y^4=z^4=(z^2)^2$.

That is, (x, y, z^2) satisfies an equation $p^4+q^4=r^2$. But we know that
 $p^4+q^4=r^2$ has no solution for positive integers from Lemma 2. Hence, a
 contradiction. Therefore, there do not exist perfect squares a, b, c st. $a^2+b^2=c^2$,
 where a, b, c are nonzero integers.

