KAIST
Math Problem of the Week

POW 4

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Problem: Cosine Matrix

Prove or disprove the following: There exists a real number 2×2 matrix M such that

$$\cos M = \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}.$$

Solution.

Claim. Given 2×2 matrix A, A^n is a linear composition of A and I_2 for all positive integer n. (I_2 is a unit 2×2 square matrix)

pf) We can verify the following equation by just calculating it.

$$A^2 - \mathrm{Tr}(A)A + \det(A)I_2 = O_2$$

Hence, A^2 is a linear composition of A and I_2.

Now, suppose that A^k is a linear composition of A and I_2. Then, A^{k+1} can be represented as a linear composition of two matrices A^2 and A, and it means A^{k+1} can be represented as a linear composition of A and I_2 since A^2 is a linear composition of both matrices. Therefore, by mathematical induction, A^n is a linear composition of A and I_2 for $n \in \mathbb{N}$.

Note that we can generalize it for $m \times m$ matrix with m independent matrices A^{m-1}, A^{m-2}, $\cdots$, I_m, by using Cayley-Hamilton theorem and induction, but I simplified it only for the 2×2 case. □

Thus, the space generated by power of A's is identical to the space generated by $\{I_2, A\}$. Let's call this space as V_A. Since cos function is defined as a limit of a sequence in V_M, and we can conclude that $\cos M$ is a linear composition of I_2 and M. Since

$$\begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix} = aI_2 + bM$$

and I_2 is diagonal, M must be a form of upper triangular matrix. Suppose that

$$M = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix},$$

then we can verify the following result

$$M^n = \begin{pmatrix} \alpha^n & \beta\left(\alpha^{n-1} + \alpha^{n-2}\gamma + \cdots + \gamma^{n-1}\right) \\ 0 & \gamma^n \end{pmatrix},$$

by mathematical induction. Then the diagonal term of $\cos M$ is $\cos\alpha = 1$ and $\cos\gamma = 1$. If $\alpha \neq \gamma$, then $\cos M$ is

$$\cos M = \begin{pmatrix} \cos\alpha & \frac{\beta}{\alpha-\gamma}(\cos\alpha - \cos\gamma) \\ 0 & \cos\gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix},$$

and if $\alpha = \gamma$, then $\cos M$ is

$$\cos M = \begin{pmatrix} \cos\alpha & \alpha\beta\sin\alpha \\ 0 & \cos\gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix},$$

both does not satisfies given equation. Thus, there does not exist 2×2 matrix M that satisfies given equation. □