POW 2022-03 Sum of vectors

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Problem 1 *For $k, n \geq 1$, let $v_1, \cdots, v_n$ be unit vectors in $\mathbb{R}^k$. Prove that we can always choose signs $\varepsilon_1, \cdots, \varepsilon_n \in \{-1, +1\}$ such that $|\sum_{i=1}^n \varepsilon_i v_i| \leq \sqrt{n}$.*

Proof. We prove it by mathematical induction. To this end, let

$$S_k = \{n \geq 1 : \forall \text{ unit vectors } \ v_1, \cdots, v_n \in \mathbb{R}^k, \ \ \exists\, \varepsilon_1, \cdots, \varepsilon_n \in \{-1, +1\} \ \text{ such that } \ \Big|\sum_{i=1}^n \varepsilon_i v_i\Big| \leq \sqrt{n}\}.$$

First $1 \in S_k$ is obvious. Now let us assume $n \in S_k$ for some $n \geq 1$ and let $v_1, \cdots, v_{n+1}$ be arbitrarily chosen unit vectors in $\mathbb{R}^k$. From the hypothesis, there exist signs $\varepsilon_1, \cdots, \varepsilon_n \in \{-1, +1\}$ such that $|w| \leq \sqrt{n}$ for $w = \sum_{i=1}^n \varepsilon_i v_i$. Since

$$|w \pm v_{n+1}|^2 = |w|^2 + |v_{n+1}|^2 \pm 2(w \cdot v_{n+1}) \leq n + 1 \pm 2(w \cdot v_{n+1}),$$

we can choose $\varepsilon_{n+1} \in \{-1, +1\}$ such that $|w + \varepsilon_{n+1} v_{n+1}|^2 \leq n + 1$. Hence $n + 1 \in S_k$ and it ends the proof by the mathematical induction. □

Proof. For $v_1, \cdots, v_n$ unit vectors in $\mathbb{R}^k$, consider a random variable

$$T(\varepsilon_1, \cdots, \varepsilon_n) = \Big|\sum_{i=1}^n \varepsilon_i v_i\Big|^2 : \{-1, 1\}^n \to [0, \infty).$$

The vector norm is computed as

$$T(\varepsilon) = \sum_{i,j} \varepsilon_i \varepsilon_j v_i \cdot v_j = \sum_{i,j} |v_i|^2 + 2 \sum_{i<j} \varepsilon_i \varepsilon_j v_i \cdot v_j = n + 2 \sum_{i<j} \varepsilon_i \varepsilon_j v_i \cdot v_j.$$

Since the **average** of T is

$$E[T] = \frac{1}{2^n} \cdot \sum_{\varepsilon \in \{-1,1\}^n} T(\varepsilon) = n,$$

there exists $\varepsilon_1, \cdots, \varepsilon_n \in \{-1, +1\}$ such that $T(\varepsilon) \leq n$ and it ends the proof. □