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<u>Def</u> (σ_i - positive word) $\sigma \in B_n$ 의 여러 표현 중에서,

σ_i를 포함하고 σ_i^{-1}과 $\sigma_j^{\pm 1}$ $(j<i)$를 포함하지 않는 표현이

단 하나라도 존재하면 σ는 σ_i - positive word이다.

그리고 그러한 표현을 σ_i - positive라 한다.

$P = \{ \sigma \in B_n \mid \exists i \in \{1, \dots, n-1\}$ s.t. σ is σ_i-positive word$\}$

를 정의하자.

<u>Lem</u> 1. $PP \subseteq P$

2. $P, \{1\}, P^{-1}$ are disjoint

3. Braid group is union of $P, \{1\}, P^{-1}$.

pf) 1. $\forall x, y \in P$, x and y is σ_i, σ_j - positive for $1 \le i, j \le n-1$

$k = \min(i,j)$라 하자. 그러면, xy는 σ_k - positive이다.

$\therefore xy \in P$

2. W.T.S: $1 \notin P$ and $\forall a \in P$, $a^{-1} \notin P$.

① $a \in P \Rightarrow a^{-1} \notin P$.

$a^{-1} = (\sigma_{i_1}^{j_1} \cdots \sigma_{i_k}^{j_k})^{-1}$ $(i_k \in \{1, \dots, n-1\}, j_k \in \{-1, 1\})$

$= \sigma_{i_k}^{-j_k} \cdots \sigma_{i_1}^{-j_1}$

Consider following two cases.

i) $\sigma_{i_1}^{j_1} \cdots \sigma_{i_k}^{j_k}$ is σ_i-positive for $i \in \{1, \ldots, n-1\}$

Let $n = \min\{i_1, \ldots, i_k\}$. Then $i = n$.

$\nexists\, \sigma_n^{-1} \in \{\sigma_{i_1}^{j_1} \cdots \sigma_{i_k}^{j_k}\}$

Consider $\sigma_{i_k}^{-j_k} \cdots \sigma_{i_1}^{-j_1}$

Since $\exists\, \sigma_n \in \{\sigma_{i_1}^{j_1} \cdots \sigma_{i_k}^{j_k}\}$ and $\sigma_n^{-1} \notin \{\sigma_{i_1}^{j_1} \cdots\}$,

$\exists\, \sigma_n^{-1} \in \{\sigma_{i_1}^{-j_1} \cdots \sigma_{i_k}^{-j_k}\}$ and $\nexists\, \sigma_n \in \{\sigma_{i_1}^{-j_1} \cdots \sigma_{i_k}^{j_k}\}$

$\therefore$ $\sigma_{i_k}^{-j_k} \cdots \sigma_{i_1}^{-j_1}$ is not σ_i-positive for

$i \in \{1, 2, \ldots, n\}$ $(\because n = \min\{i_1, \ldots, i_k\}$

ii) $\sigma_{i_1}^{j_1} \cdots \sigma_{i_k}^{j_k}$ is not σ_i-positive for $\forall i \in \{1, 2, \ldots, n-1\}$

Let $n = \min\{i_1, \ldots, i_k\}$. Then, $\exists\, \sigma_n^{-1} \in \{\sigma_{i_1}^{j_1} \cdots \sigma_{i_k}^{j_k}\}$

Consider $\sigma_{i_k}^{-j_k} \cdots \sigma_{i_1}^{-j_1}$.

$\exists\, \sigma_n, \sigma_n^{-1} \in \{\sigma_{i_k}^{-j_k} \cdots \sigma_{i_1}^{-j_1}\}$ since $\exists\, \sigma_n, \sigma_n^{-1}$

$\in \{\sigma_{i_1}^{j_1} \cdots \sigma_{i_k}^{j_k}\}$

$\therefore$ By i), ii). a^{-1} is not σ_i-positive for any of its expression

② $1 \notin P$

Assume $1 \in P$. Then $1^{-1} = 1 \notin P$ by ①.

contradiction. Thus, $1 \notin P$

3. σ_i 사이의 연산이 복잡하므로 permutation matrix와 비슷한 행렬로 나타내자.

$\sigma_i = D_i(a)D_{i+1}(b)T_{i\,i+1}$ ($a, b > 1$, $D_i(a)$ and $T_{i\,i+1}$: $n \times n$ elementary row operation matrix)

① $\det(\sigma_i^k) = (-1)^k a^k b^k \neq 1$ for $\forall k > 1$.

$\therefore$ σ_i의 order는 ∞ 이다.

② $\sigma_i \sigma_j = D_i(a)D_{i+1}(b)T_{i\,i+1}\, D_j(a)D_{j+1}(b)T_{j\,j+1}$

disjoint

disjoint

$= D_j(a)D_{j+1}(b)T_{j\,j+1}D_i(a)D_{i+1}(b)T_{i\,i+1}$

$= \sigma_j \sigma_i$

$(|i-j| \geq 2)$

③ $\sigma_i\sigma_{i+1}\sigma_i = D_i(a)D_{i+1}(b)T_{i\,i+1}D_{i+1}(a)D_{i+2}(b)T_{i+1\,i+2}D_i(a)D_{i+1}(b)T_{i\,i+1}$

$= D_i(a^2)D_{i+1}(ab)D_{i+2}(b^2)T_{i\,i+2}$

$\sigma_{i+1}\sigma_i\sigma_{i+1} = D_{i+1}(a)D_{i+2}(b)T_{i+1\,i+2}D_i(a)D_{i+1}(b)T_{i\,i+1}D_{i+1}(a)D_{i+2}(b)T_{i+1\,i+2}$

$= D_i(a^2)D_{i+1}(ab)D_{i+2}(b^2)T_{i\,i+2}$

Thus, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

P_σ를 permutation component 라 정의하자.

$\sigma \in B_n$은 $\sigma = D_1(a^{m_1}b^{q_1}) \cdots D_n(a^{m_n}b^{q_n})P_\sigma$

(P_σ: permutation matrix)로 표기할 수 있다. 그리고 이는

유일한 표기법이다. ($\sum m_i = \sum q_i$) ($\sigma_1 \cdots \sigma_{n-1}$에 대해서 $\sum m_i = \sum q_i$

$\to \forall B_n$로 $\sum m_i = \sum q_i$이다.)

<u>Lem.</u> $[V_e] = \{\sigma \in B_n \mid P_\sigma = I_n\}$

$\trianglelefteq B_n$

pf. $\forall g \in B_n$, $\forall a \in [V_e]$

$$gag^{-1} = \overbrace{D_1(a^{m_1'}b^{q_1'}) \cdots D_n(a^{m_n'}b^{q_n'})}^{\text{diagonal}} P_\sigma \cdot D_1(a^{m_1}b^{q_1}) \cdots D_n(a^{m_n}b^{q_n}) \cdot I_n \, P_\sigma^{-1} \underbrace{D_n(a^{-m_n'}b^{-q_n'}) \cdots D_1(a^{-m_1'}b^{-q_1'})}_{\text{diagonal}}$$

diagonal

$\in [V_e]$

$\therefore g[V_e]g^{-1} \subseteq [V_e]$.

$e \in [V_e]$. $\forall a, b \in [V_e]$, $ab^{-1} \in [V_e]$

diagonal diagonal

Thus, $[V_e] \trianglelefteq B_n$.

<u>Lem</u>. $[0] = \{ \sigma \in B_n \mid |\det(\sigma)| = 1 \} \trianglelefteq B_n$

$\underset{\hookrightarrow \text{matrix}}{\sigma}$

pf. $e \in [0]$. $\forall a, b \in B_n$, $\det(ab^{-1}) = \det(a)\det(b)^{-1} = 1$

$\therefore ab^{-1} \in B_n$.

$[0] \leq B_n$.

$\forall g \in B_n$ $\forall a \in [0]$, $\det(gag^{-1}) = \det(g)\det(a)\det(g)^{-1}$

$= \det(g) \cdot \det(g)^{-1} = 1$

Thus, $gag^{-1} \in [0]$.

$\therefore [0] \trianglelefteq B_n$.

<u>Def</u>. (reduced representation)

$\forall \sigma \in B_n$ 에 대해 $(n \geq 3)$

$\sigma =$ (reduced decomposition of σ when $\sigma_i^2 = e$ for $1 \leq i \leq n-1$)

$\sigma_{n-1}^{k} (\sigma_1^2 \sigma_2^2)^{m_1} \cdots (\sigma_{n-2}^2 \sigma_{n-1}^2)^{m_{n-2}}$ $(m_i, k \in \mathbb{Z}, 1 \leq i \leq n-2)$

로 표현할 수 있다.

pf. $B_n/[0] \cong \mathbb{Z}$ $\quad B_n = \{ \cdots \sigma_{n-1}^{-1}[0], e[0], \sigma_{n-1}[0] \cdots \}$

$B_n/[V_e] \cong S_n$ $\quad B_n = \{ a_\lambda [V_e] \mid \lambda \in S_n \}$ (a_λ: $P_\sigma = \lambda$인 σ)

$K = [0] \cap [V_e] \trianglelefteq B_n$

$B_n/K \cong B_n/[V_e] \times B_n/[0]$ — (1)

W.T.S : K는 finite dimensional vector space이고

basis는 $\{\sigma_1^2\sigma_2^2, \cdots \sigma_{n-2}^2\sigma_{n-1}^2\}$ 이다.

pf. $\forall c = D_1(a^{m_1}b^{q_1}) \cdots D_n(a^{m_n}b^{q_n})$

$d = D_1(a^{m_1'}b^{q_1'}) \cdots D_n(a^{m_n'}b^{q_n'})$

$r \in \mathbb{Z}$

i) $c \cdot d = D_1(a^{m_1+m_1'}b^{q_1+q_1'}) \cdots D_n(a^{m_n+m_n'}b^{q_n+q_n'})$

$\in K \quad (\sum m_i + \sum m_i' = \sum q_i + \sum q_i' = 0)$

ii) $c^r = D_1(a^{rm_1}b^{rq_1}) \cdots D_n(a^{rm_n}b^{rq_n})$

$\in K \quad (\sum rm_i = \sum rq_i = 0)$

commutativity, associativity, distributive 로 성립.

$\sum_{i=1}^{n-1} m_i = \sum_{i=1}^{n-1} q_i = 0$ 이므로 $\dim K = n-2$.

K은 basis는 $\{\sigma_1^2\sigma_2^2 \cdots \sigma_{n-2}^2\sigma_{n-1}^2\}$ 이다.

: diagonal : diagonal

$\therefore \forall d \in K,\ d = (\sigma_1^2\sigma_2^2)^{m_1} \cdots (\sigma_{n-2}^2\sigma_{n-1}^2)^{m_{n-2}}$

for $m_1 \cdots m_{n-2} \in \mathbb{Z}$

$\therefore \forall \sigma \in B_n,\ \sigma = \lambda \cdot \sigma_{n-1}^k \cdot k\ (k \in K)$ (by (1))

$= \lambda\, \sigma_{n-1}^k (\sigma_1^2\sigma_2^2)^{m_1} \cdots (\sigma_{n-2}^2\sigma_{n-1}^2)^{m_{n-2}}$

로 나타낼 수 있다.

이때, λ에 대해 조금 더 고민해볼 수 있다. λ의 per. comp.를 고민했을 때,

$inv(\lambda)$ (= # of $(i<j)$ s.t. $(\lambda(i)>\lambda(j))$)

개의 generator의 곱으로 λ을 표현할 수 있다.

그리고, 그 generator들의 index($1 \sim n-1$)의 minimum

값은, a의 matrix representation인 P에 대해서

$\max\{i \mid P[i][i]=1\}$ 이다.

이 generator의 곱은 $\sigma_i^2=e$로 정해서 $\sigma_1 \sim \sigma_{n-1}$을

S_n의 원소로 고려하였을 때 구성할 수 있다.

(e.g. $\sigma=\sigma_1^2\sigma_2^2\sigma_3$, $\lambda=e\cdot e\cdot\sigma_3=\sigma_3$)

$\therefore$ $\sigma \in B_n$에 대해서, $\sigma=$ (reduced decomposition of σ

considering it as an element of S_n) $\cdot\ \sigma_{n-1}^k\ \cdot$

$(\sigma_1^2\sigma_2^2)^{m_1}\cdots(\sigma_{n-2}^2\sigma_{n-1}^2)^{m_{n-2}}$.

W.T.S : $\forall a \neq 1$, $a \in P$ or $a^{-1} \in P$.

pf. If $a \in P$ (o)

Assume $a \notin P$.

W.T.S : $a^{-1} \in P$.

a 의 reduced decomposition에 대해 생각해보자.

다만. reduced decomposition의 형태를 조금

달리해보자

$$a = (\sigma_{i_1} \cdots \sigma_{i_t})\, \sigma_{n-1}^{k} (\sigma_1^2 \sigma_2^2)^{m_1} \cdots (\sigma_{n-2}^2 \sigma_{n-1}^2)^{m_{n-2}}$$

$l = \max\{i : [P]_{ii} = 1\}$에 대해 생각해보자.

$m_1' \cdots m_{l-1}' = 0$이 되도록 설정할 수 있다.

그리고 $i_1 \cdots i_t$ 중에 l이 단 하나만 사용되도록

할 수도 있다.

이와 별개로, $\min_{1 \le p \le t} m_p = l$이 되게 된다.

$$\therefore a = (\sigma_{i_1'} \cdots \sigma_{i_t'})\, \sigma_{n-1}^{k'} (\sigma_l^2 \sigma_{l+1}^2)^{m_l'} \cdots (\sigma_{n-2}^2 \sigma_{n-1}^2)^{m_{n-2}'}$$

(만약 $l=n$ 이라면, $a=1$ 이 된다. contradiction)
$l=n-1$ 은 될 수 없다. 따라서 $l \le n-2$.

$\therefore a = (\sigma_{i_1} \cdots \sigma_{i_t})\, \sigma_{n-1}^{k''} (\sigma_l^2 \sigma_{l+1}^2)^{m_l''} \cdots (\sigma_{n-2}^2 \sigma_{n-2}^2)^{m_{n-2}''}$

$\min_{1\le p\le t} j_p = n$

$\min\{i_1 \cdots i_t, n-1, l, l+1 \cdots n-2\} =$

$m_{l''} \neq 0$ 이므로 $m_{l''} < 0$ 이다. ($\because a \notin P$, $\exists l \in \{i_1, \cdots, i_t\}$)

$a^{-1} = (\sigma_l^2 \sigma_{l+1}^2)^{-m_l''} \cdots (\sigma_{n-2}^2 \sigma_{n-1}^2)^{-m_{n-2}''}\, \sigma_{n-1}^{-k''}$
$\boxed{(\sigma_{i_t}^{-1} \cdots \sigma_{i_1}^{-1})}$
$= (\sigma_l^2 \sigma_{l+1}^2)^{-m_l'''} \cdots (\sigma_{n-2}^2 \sigma_{n-1}^2)^{-m_{n-2}'''}\, \sigma_{n-1}^{-k''}$
$\boxed{(\sigma_{i_t}^{-1} \cdots \sigma_l^{1} \cdots \sigma_{i_1}^{-1})}$

$\sigma_l^{-1} \to \sigma_l$

$aa^{-1} = (\sigma_l^2 \sigma_{l+1}^2)^{-m_l'''} \cdots (\sigma_{n-2}^2 \sigma_{n-1}^2)^{-m_{n-2}'''}\, \sigma_{n-1}^{-k''}$
$(\sigma_{i_t}^{-1} \cdots \sigma_l \cdots \sigma_{i_1}^{-1}) \cdot (\sigma_{i_1} \cdots \sigma_l \cdots \sigma_{i_t})$
$\sigma_{n-1}^{k''} (\sigma_l^2 \sigma_{l+1}^2)^{m_l''} \cdots (\sigma_{n-2}^2 \sigma_{n-1}^2)^{m_{n-2}''}$

$= (\sigma_l^2 \sigma_{l+1}^2)^{-m_l'''} \cdots (\sigma_{n-2}^2 \sigma_{n-1}^2)^{-m_{n-2}'''}$.
$\begin{pmatrix} \ddots & & \\ & ab & \\ & & ab \ddots \end{pmatrix} (\sigma_l^2 \sigma_{l+1}^2)^{m_l''} \cdots (\sigma_{n-2}^2 \sigma_{n-1}^2)^{m_{n-2}''}$

$\therefore \begin{pmatrix} ab^{-m_l'''} & & \\ & \ddots & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \ddots & & \\ & ab & \\ & & ab \ddots \end{pmatrix} = \begin{pmatrix} ab^{-m_l''} & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$

$\therefore\ (ab)^{-me'''}\cdot(ab) = (ab)^{-me''}$ or — (1)

$(ab)^{-me'''}\cdot 1 = (ab)^{-me''}$ — (2)

(1) : $-me''' = -me'' - 1 \geq 0$

(2) : $-me''' = -me'' > 0$

$\therefore\ -me'' \geq 0$

$\therefore$ σ_ℓ 과 관련된 항은 $(\sigma_\ell^2\sigma_{\ell+1}^2)^{-me'''}$ 과 σ_ℓ 인데

그 중, $(\sigma_\ell^2)^{-me'''}$ 에서 $-2me''' \geq 0$ 이므로

σ_ℓ 과 관련된 항 중 σ_ℓ^{-1} 을 가진 건 없다.

$\therefore\ a^{-1} = (\sigma_\ell^2\sigma_{\ell+1}^2)^{-me'''} \cdots (\sigma_{n-2}^2\sigma_{n-1}^2)^{-m_{n-2}'''}\sigma_{n-1}^{-k''}$

$(\sigma_{i}^{-1} \cdots \sigma_\ell \cdots \sigma_{i_1}^{-1})$.

a^{-1}은 σ_ℓ- positive 하다.

따라서 $a^{-1} \in P$.

Def. (left-invarient total order on B_n, a.k.a Dehornoy order)

$\forall a, b \in B_n$, $a < b \iff b^{-1}a \in P$
$a = b \iff b^{-1}a = 1$

pf. ① $a^{-1}a = 1$. $(a \in B_n)$

$\therefore a \leq a$ ■

② If $a < b$, $b < c$ $(a, b, c \in B_n)$

$b^{-1}a \in P$ and $c^{-1}b \in P$,

Since $PP \subseteq P$, $(c^{-1}b)(b^{-1}a) = (c^{-1}a) \in P$

$\therefore a < c$ ■

(If $a = b$ or $b = c$, it can be proven too.
since $1 \cdot P = P \cdot 1 = P$)

③ If $a \leq b$, $b \leq a$ $(a, b \in B_n)$

$b^{-1}a \in \{1\} \cup P$ and $a^{-1}b \in \{1\} \cup P$.

If $b^{-1}a \notin \{1\}$, $b^{-1}a \in P$.

$\therefore (b^{-1}a)^{-1} = a^{-1}b \notin P$. and $a^{-1}b \notin \{1\}$

contradiction

$\therefore b^{-1}a = 1$. $a = b$ ■

④ W.T.S : $a \le b$ or $b \le a$ $(a, b \in B_n)$

$b^{-1}a \in P$ or $b^{-1}a \in 1$ or $b^{-1}a \in P^{-1}$ by lemma

If $b^{-1}a \in P$, $a \le b$.

If $b^{-1}a = 1$, $a = b$

If $b^{-1}a \in P^{-1}$, $a^{-1}b \in P$. $b \le a$

⑤ If $a < b$, $ca < cb$ $(a, b, c \in P)$

$b^{-1}a \in P$.

$(cb)^{-1}(ca) = b^{-1}c^{-1}ca = b^{-1}a \in P$

Thus. $ca < cb$