KAIST POW 2021-24 The squares of wins and losses

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There are n people participating to a chess tournament and every two players play one game. There are no draws. Let a_i be the number of wins of the i-th player and b_i be the number of losses of the i-th player. Prove that $\sum_{i\in[n]} a_i^2 = \sum_{i\in[n]} b_i^2$.

Preliminaries

Proposition 1. Every player participates in $n-1$ matches, hence $a_i + b_i = n-1$.

Proposition 2. There are $n(n-1)/2$ matches, and each match creates a loser and a winner, hence $\sum_{i\in[n]} a_i = \sum_{i\in[n]} b_i = n(n-1)/2$.

Solution 1 (soulless)

If we write it down:

$\sum_{i\in[n]} a_i^2 = \sum_{i\in[n]} b_i^2$
$\sum_{i\in[n]} a_i^2 = \sum_{i\in[n]}(n-1-a_i)^2$. (Prop 1)
$\sum_{i\in[n]} a_i^2 = n(n-1)^2 - \sum_{i\in[n]}(2n-2)a_i + \sum_{i\in[n]} a_i^2$. (Expansion)
$\sum_{i\in[n]}(2n-2)a_i = n(n-1)^2$ (Transposition)
$(2n-2)n(n-1)/2 = n(n-1)^2$ (Prop 2)
$n(n-1)^2 = n(n-1)^2$
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Solution 2 (soulful)

Let's prove $\sum_{i\in[n]}(a_i^2 - a_i) = \sum_{i\in[n]}(b_i^2 - b_i)$ instead, since this is equivalent to the original problem by Proposition 2.

Then, the quantity in LHS is equal to the number of ordered triplets of person i, j, k such that

- i beats j (hence $i \neq j$)
- i beats k (hence $i \neq k$)

- $j \neq k$

Consequently, the quantity in RHS is equal to the number of ordered triplets of person i, j, k such that

- i is beaten by j (hence $i \neq j$)
- i is beaten by k (hence $i \neq k$)
- $j \neq k$

Let's enumerate all unordered triplets of person i, j, k, and check how many order (among all $3!$ orders) contributes to the both quantity.

There are two cases:

- In the first case, the competition result between person i, j, k are cyclic. WLOG, suppose that i beats j, j beats k, k beats i. In this case, nobody is beaten by both, so the triplet neither contributes to LHS nor RHS.
- In the second case, the competition result between person i, j, k are acyclic. If we consider a directed graph of three vertices where there is an edge from a to b if a beats b, then the graph is acyclic, which means it admits a topological ordering. Since every pair between $\{i, j, k\}$ is compared it corresponds to the total ordering between those three people. WLOG, suppose that i beats j, j beats k, i beats k. Then i beats both, and k is beaten by both, and j beats one and is beaten by one. There are exactly two triplet that contributes to LHS (i, j, k, i, k, j) and exactly two triplet that contributes to RHS (k, i, j, k, j, i)

Thus, every ordered triplet of person i, j, k contributes to LHS and RHS by a same amount. As a result, LHS = RHS.