POW 2021-21: Different Unions

Problem. Let F be a family of nonempty subsets of $[n] = \{1, \cdots, n\}$ such that no two disjoint subsets of F have the same union. In other words, for $F = \{A_1, A_2, \cdots, A_k\}$, there exists no two sets $I, J \subseteq [k]$ with $I \cap J = \emptyset$ and $\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j$. Determine the maximum possible size of F.

Solution. Suppose that F has size $k > n$.

Let us define the ***binary vector form*** of $A \subset [n]$ as a vector $v \in \{0,1\}^n$ such that the i^{th} element of v is 1 if $i \in A$, and 0 otherwise. (*For instance, the vector form of* $\{1,2,4\} \in [5]$ *becomes* $(1,1,0,1,0)$.)

Now let us define a binary matrix $M \in \{0,1\}^{k \times n}$ such that each row $v_i \in \{0,1\}^n$ is a vector form of the set $A_i \in F$ for $i = 1, \cdots, k$. Since $k > n$, the matrix M must have rank at most n. Therefore there exists $c_i \in \mathbb{R}$ for $i = 1, \cdots, k$ such that:

$$\sum_{i=1}^{k} c_i v_i = \mathbf{0}.$$

Now we can eliminate weights $c_i = 0$ (*if any*) and move terms s.t. $c_i < 0$ to the RHS to rewrite as:

$$u := \sum_{S_1} c_i' v_i = \sum_{S_2} c_i' v_i, \tag{1}$$

where $S_1 := \{i : c_i > 0\}$, $S_2 := \{i : c_i < 0\}$, $c_i' = c_i$ for $i \in S_1$ and $c_i' = -c_i$ for $i \in S_2$.

Note that $S_1 \cap S_2 = \emptyset$ by definition. Also, note that $c_i' > 0$ for all $i \in S_1 \cup S_2$ (*which is rigorously okay since F contains 'nonempty' subsets of* $[n]$, *and hence no v_i is a zero vector*). Since $v_i \in \{0,1\}^n$ for all $i \in k$ and $c_i' > 0$ for all $i \in S_1 \cup S_2$, the vector $u \in \mathbb{R}^n$ defined in (1) must have nonnegative elements.

Let us denote the set of indices of nonnegative entries of u as U. From $u := \sum_{S_1} c_i' v_i$ and $c_i' > 0$, observe that:

$$\begin{aligned}
\ell \in U \quad & \Leftrightarrow \quad u_\ell > 0 \text{ for some } \ell = [n] \\
& \Leftrightarrow \quad \exists \text{ (\textit{at least one}) } i \in S_1 \text{ s.t. } v_i \text{ has } \ell^{\text{th}} \text{ element } 1 \\
& \Leftrightarrow \quad \exists \text{ (\textit{at least one}) } i \in S_1 \text{ s.t. } \ell \in A_i \\
& \Leftrightarrow \quad \ell \in \bigcup_{i \in S_1} A_i
\end{aligned}$$

and hence $U = \bigcup_{i \in S_1} A_i$. Similarly, from $u := \sum_{S_2} c_i' v_i$ and $c_i' > 0$, we also have $U = \bigcup_{j \in S_2} A_j$, and therefore $\bigcup_{i \in S_1} A_i = \bigcup_{j \in S_2} A_j$ for $S_1 \cap S_2 = \emptyset$. Therefore no possible cases exist for $k > n$.

Finally, we can always find F of size $k = n$, which is $F_{\text{max}} = \{\{1\}, \{2\}, \cdots, \{n\}\}$. It is obvious that no two disjoint subsets of F_{max} have the same union, since W.L.O.G. if we label $A_i = \{i\}$ for $i = 1, \cdots, n$, then we have

$$\bigcup_{i \in I} A_i = I \neq J = \bigcup_{j \in J} A_j$$

for all $I, J \subseteq [n]$ with $I \cap J = \emptyset$. Therefore the maximum possible size of F is $\boxed{n}$.