POW 2021-22 Sum of Fractions

2021xxxx 조정휘(대학원생)

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Problem. Determine all rational numbers that can be written as

$$\frac{1}{n_1} + \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} + \cdots + \frac{1}{n_1 n_2 n_3 \cdots n_k},$$

where $n_1, n_2, n_3, \cdots, n_k$ are positive integers greater than 1.

Answer: All rationals between 0 and 1.

From

$$0 < \frac{1}{n_1} + \frac{1}{n_1 n_2} + \cdots + \frac{1}{n_1 n_2 \cdots n_k} \leq \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} < 1,$$

it is obvious that the result lies in $(0, 1)$. Conversely, consider any rational $q_1 = a/b$ with two integers $0 < a < b$. We can find the smallest integer $n_1 > 1$ with $n_1 a \geq b$. Then we have

$$q_1 = \frac{1}{n_1}\left(1 + \frac{n_1 a - b}{b}\right).$$

Since $n_1 a - b = (n_1 - 1)a + a - b < b + a - b = a$, the quotient $q_2 = (n_1 a - b)/b$ is either zero or another rational between 0 and 1. If $q_2 = 0$, terminate here. Otherwise, repeat the same process again for q_2 to yield n_2 and q_3, and so on. The process eventually stops because, by comparison on the numerator, one sees that always $q_n > q_{n+1}$ with the difference being no less than $1/b$. As a result, one has

$$q_1 = \frac{1}{n_1}\left(1 + \frac{1}{n_2}\left(1 + \frac{1}{n_3} \cdots \left(1 + \frac{1}{n_k}\right) \cdots\right)\right)$$

for some integer k. Multiplying the terms out, we get the desired result.