# POW 2021-21 Different unions

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## *Problem*

Let F be a family of nonempty subsets of [n] ={1,…,n} such that no two disjoint sub-sets of F have the same union. Determine the maximum possible size of F.

## *Sol*

Let $\mathrm{F}_1$ be a family of nonempty subsets of [n] s.t $\{1\} \subseteq \mathrm{s}$ for $\forall \mathrm{s} \in \mathrm{F}_1$ and $\mathrm{F}_1^c = \mathrm{P}([\mathrm{n}]) - \mathrm{F}_1 - \{\phi\}$

(P([n])={A | A⊆[n]} (power set of [n]), $\mathrm{F}_1 \cap \mathrm{F}_1^c = \phi$, $|\mathrm{F}_1| = 2^{n-1}$, $|\mathrm{F}_1^c| = 2^{n-1} - 1$)

Ex) for [5]={1,2,3,4,5},

$$\mathrm{F}_1 = \{\, \{1,2,3,4,5\},$$
$$\{1,3,4,5\}, \{1,2,4,5\}, \{1,2,3,5\}, \{1,2,3,4\},$$
$$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\},$$
$$\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\},$$
$$\{1\}\,\}$$
$$\mathrm{F}_1^c = \{\, \{2,3,4,5\},$$
$$\{3,4,5\}, \{2,4,5\}, \{2,3,5\}, \{2,3,4\},$$
$$\{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\},$$
$$\{2\}, \{3\}, \{4\}, \{5\}\}$$

Since F is a family of nonempty subsets of [n], F is union of $\mathrm{f}_1$ and $\mathrm{f}_1^c$ s.t $\mathrm{f}_1$ and $\mathrm{f}_1^c$ are subset of $\mathrm{F}_1$ and $\mathrm{F}_1^c$ respectively, i.e., $\mathrm{F} = \mathrm{f}_1 \cup \mathrm{f}_1^c$ s.t $\mathrm{f}_1 \subseteq \mathrm{F}_1$ and $\mathrm{f}_1^c \subseteq \mathrm{F}_1^c$.

Since $([\mathrm{n}] - \mathrm{s}) \in \mathrm{F}_1^c$ and $([\mathrm{n}] - \mathrm{s}) \cup \mathrm{s} = [\mathrm{n}]$ for $\forall \mathrm{s} \in \mathrm{f}_1$, only one element of set O( $=\{[\mathrm{n}] - \mathrm{s} \mid ([\mathrm{n}] - \mathrm{s}) \in \mathrm{F}_1^c$ for $\forall \mathrm{s} \in \mathrm{f}_1\}$ ) can be element of $\mathrm{f}_1^c$. If two elements of O are element of $\mathrm{f}_1^c$, there are same union $[\mathrm{n}](= ([\mathrm{n}] - s_1) \cup s_1 = ([\mathrm{n}] - s_2) \cup s_2)$ for $s_1, s_2 \in O$. So, $\mathrm{f}_1^c$ consists of one element of O and subset of $\mathrm{F}_1^c - O$.

Therefore,

$$\mathrm{F} = \mathrm{f}_1 \cup \mathrm{f}_1^c \subseteq \mathrm{f}_1 \cup \big( (\text{one element of O}) \cup (\mathrm{F}_1^c - O) \big) \rightarrow |F| \le |\mathrm{f}_1| + 1 + (2^{n-1} - 1) - |\mathrm{f}_1| = 2^{n-1}$$

$(|\mathrm{f}_1| \le 2^{n-1} - 1)$

When $|\mathrm{f}_1| = 2^{n-1}$, i.e., $\mathrm{f}_1 = \mathrm{F}_1$, $\mathrm{F} \subseteq \mathrm{f}_1 \cup (\text{one element of O}') = \mathrm{F}_1 \cup (\text{one element of } \mathrm{F}_1^c)$. and by same reason above, only none element of O′ can be possible.

( $\mathrm{O}' = \{\, [\mathrm{n}] - \mathrm{s} \mid ([\mathrm{n}] - \mathrm{s}) \in \mathrm{F}_1^c$ for $\forall \mathrm{s} \in (\mathrm{F}_1 - \{[n]\})\,\} = \mathrm{F}_1^c$ )

we can make maximum possible size of F by adding one element of $O'$ to $F_1$

$(i.e., F = F_1 \cup (\text{one element of } F_1^c))$

It can be proved easily that no two disjoint sub-sets of F have the same union. Two disjoint sub-sets of F can be possible only $s \in F_1$ and one element of $F_1^c$ respectively. So all union of $s \in F_1$ and one element of $F_1^c$ are different each other. Conclusively,

$$\therefore \max(F) = 2^{n-1} + 1$$