POW 2021-20 A circle of perfect squares

전해구(기계공학과 졸업생)

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Problem

Say a natural number n is a cyclically perfect if one can arrange the numbers from 1 to n on the circle without a repeat so that the sum of any two consecutive numbers is a perfect square. Show that 32 is the smallest cyclically perfect number. Find the second smallest cyclically perfect number.

Sol

Cyclically perfect number n means that there exist at least two different numbers a&b besides i s.t a+i and b+i are perfect square for i=1,2,…,n.

For n = 1,2,3, we can show easily n is not a cyclically perfect.

For n = 4, we need to at least 12&5 for 4 to be a cyclically perfect. For n = 12, we need to at least 17&1 for 8. For n = 17, we need to 19&8. For n = 19, we need to 31&17 for 18. Since use number 18 twice to make 36(=18+18), we use next perfect square 49 to be a cyclically perfect.

Therefore n is greater than 31 to be a cyclically perfect.

Let see whether 31 is a cyclically perfect or not. I list the a&b,i s.t a+i and b+i are perfect square and a,b are unique for i.

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<th>a</th>
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</table>

By using list,

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for [19 – 30 – 6] of the list, left side of 19 can be 17 or 6. But since 6 is already used, left side of 19 is 17.

By using list,


So, we have to arrange the following sequence to make n a cyclically perfect.


Left side of 5 can be 11 or 4.

**Case1) left side of 5 is 11**

By using list,


Side of 14 can be 2,11,22 but 11 is already used. So,


Right side of 2 can be 7,14,23. But 7,14 are already used. By using list and previous fact,


Side of 4 and 13 can be respectively 21,12,5 and 23,12,3. But 5 and 23 is already used. So,

$$[21–6], [4], [12], [13], [3] → [6 – ~ – 21 – 4 – 12 – 13 – 3]$$

Left side of 6 can be 3,10,19. But 3 and 19 are already used. So,


And by using last [1],[15],


Or


But those sequences can’t make n be a cyclic number.

**Case2) left side of 5 is 4**


Left side of 4 can be 21 or 12.
(1) Left side of 4 is 21

\[ 4 - 5 \sim \sim - 22 \rightarrow [6 - 30 - 19 - 17 - 8 - 28 - 21 - 4 - 5 \sim \sim - 22] \rightarrow [6 \sim \sim - 22] \]

Side of 15 can be 21,10,1. But 21 is already used. so by using list and previous fact,

\[ 1 - 15 - 10 - 26 - 23 \]

Side of 12 can be 4,13,24. But 4 is already used. so by using list and previous fact,

\[ 13 - 12 - 24 - 25 - 11 \]

left side of 6 can be 10 or 3. But 10 is already used so,

\[ 6 - \sim - 22 \rightarrow [3 - 6 - \sim - 22] \]

Side of 22 can be 14 or 3. But 3 is already used. so, right side of 22 is 14

\[ 3 - 6 - \sim - 22 \rightarrow [3 - 6 - \sim - 22 - 14] \]

Left Side of 1 can be 24,15,8. But 24,15 is already used. so,

\[ 1 - 15 - 10 - 26 - 23 \rightarrow [14 - 22 - \sim - 6 - 3 - 1 - 15 - 10 - 26 - 23] \]

Sequences remain like following:

\[ [14 - 22 - \sim - 6 - 3 - 1 - 15 - 10 - 26 - 23], [13 - 12 - 24 - 25 - 11], [2] \]

It can arrange like following

\[ [2 - 14 - 22 - \sim - 6 - 3 - 1 - 15 - 10 - 26 - 23 - 13 - 12 - 24 - 25 - 11] \]

or

\[ [13 - 12 - 24 - 25 - 11 - 14 - 22 - \sim - 6 - 3 - 1 - 15 - 10 - 26 - 23 - 2] \]

But these sequences can’t make n be a cyclically number.

(2) Left side of 4 is 12

\[ 4 - 5 - 31 - 18 - 7 - 29 - 20 - 16 - 9 - 27 - 22 \rightarrow [12 - 4 - 5 \sim \sim - 22] \]

Left side of 12 can be 24 or 13

1) left side of 12 is 13

\[ 12 - 4 - 5 \sim \sim - 22 \rightarrow [13 - 12 - 4 - 5 \sim \sim - 22] \]

Left side of 13 can be 23 or 3

First, check when left side of 13 is 23

\[ [13 - 12 - 4 - 5 \sim \sim - 22] \rightarrow [23 - 13 - 12 - 4 - 5 \sim \sim - 22] \]

By using list,

\[ [23 - 13 - 12 - 4 - 5 \sim \sim - 22] \rightarrow [10 - 26 - 23 - 13 - 12 - 4 - 5 \sim \sim - 22] \]

Side of 21 can be 15 or 4. But 4 is already used. so,
For the [24 – 25 – 11], Side of 24 and 11 can be respectively 12 or 1 and 14 or 5. But 12 and 5 are already used. so,


Sequences remain like following


It can arrange like following;


or


or


Or


But these sequences can’t make n be a cyclically number.

Second, check when left side of 13 is 3


Left side of 23 can be 2 or 13 but 13 is already used. so,


Left side of 2 can be 7,14,23. But 7,23 is already used. so,


Right side of 22 can be 3,14. But only 14 can be possible. So,


Left side of 11 can be 5 or 14 in the list. But 5 and 14 is already used. so, it’s contradiction.

2) left side of 12 is 24


Left side of 11 can be 5,14. But 5 is already used. by using list and previous fact,


Left side of 14 can be 2,22. But only 2 can be possible and side of 2 can be 7,14,23. But 7 is already used.
By using list and previous fact,
\[14 - 11 - 25 - 24 - \sim - 22\] → \[10 - 26 - 23 - 2 - 14 - 11 - 25 - 24 - \sim - 22\] \hspace{1cm} \ldots \hspace{1cm} (a)

Side of 13 can be 23,12,3. But 12 is already used. and by using list,
\[23 - 13 - 3\] → \[10 - 26 - 23 - 13 - 3\]

But this sequence is contradicted to sequence (a).

Conclusively, \(n=31\) can't be a cyclic number.

I can find sequences s.t \(n=32,33\) is a cyclically number by using similar way and correcting list.
Ex) \((a,b)=(4,32,17)\).
But I don’t know how many different sequences that make \(n=32,33\) a cyclically number. The sequences that I find is following;
\[ \pi = 3.14 \]