POW 2021-20 A circle of perfect squares

전해구(기계공학과 졸업생)

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Problem

Say a natural number n is a cyclically perfect if one can arrange the numbers from 1 to n on the circle without a repeat so that the sum of any two consecutive numbers is a perfect square. Show that 32 is the smallest cyclically perfect number. Find the second smallest cyclically perfect number.

Sol

Cyclically perfect number n means that there exist at least two different numbers a&b besides i s.t a+i and b+i are perfect square for i=1,2,…,n.

For $n = 1,2,3$, we can show easily n is not a cyclically perfect.

For $n = 4$, we need to at least 12&5 for 4 to be a cyclically perfect. For $n = 12$, we need to at least 17&1 for 8. For $n = 17$, we need to 19&8. For $n = 19$, we need to 31&17 for 18. Since use number 18 twice to make 36(=18+18), we use next perfect square 49 to be a cyclically perfect.

Therefore n is greater than 31 to be a cyclically perfect.

Let see whether 31 is a cyclically perfect or not. I list the a&b,i s.t a+i and b+i are perfect square and a,b are unique for i.

a	i	b
18	31	5
19	30	6
20	29	7
21	28	8
22	27	9
23	26	10
24	25	11
31	18	7
19	17	8
20	16	9

<List 1>

By using list,

$[5-31-18] \rightarrow [5-31-18-7] \rightarrow [5-31-18-7-29-20]$. Right side of 20 can be 16 or 5. But since 5 is already used, right side of 20 is 16. So, $[5-31-18-7-29-20-16]$.

By using list,

$$[5-31-18-7-29-20-16] \quad \rightarrow \quad [5-31-18-7-29-20-16-9-27-22]$$

for $[19-30-6]$ of the list, left side of 19 can be 17 or 6. But since 6 is already used, left side of 19 is 17.

By using list,

$[19-30-6] \rightarrow [21-28-8-17-19-30-6]$

So, we have to arrange the following sequence to make n a cyclically perfect.

$[5-31-18-7-29-20-16-9-27-22], [21-28-8-17-19-30-6], [23-26-10], [24-25-11], [1], [2], [3], [4], [12], [13], [14], [15]$

Left side of 5 can be 11 or 4.

Case1) left side of 5 is 11

By using list,

$[5-31-18-7-29-20-16-9-27-22] \rightarrow [24-25-11-5-31-18-7-29-20-16-9-27-22]$

Side of 14 can be 2,11,22 but 11 is already used. So,

$[24-25-\sim-27-22] \rightarrow [24-25-\sim-27-22-14-2]$

Right Side of 2 can be 7,14,23. But 7,14 are already used. By using list and previous fact,

$[24-25-\sim-27-22-14-2] \rightarrow [24-25-\sim-27-22-14-2-23-26-10]$

Side of 4 and 13 can be respectively 21,12,5 and 23,12,3. But 5 and 23 is already used. So,

$[21\sim6], [4], [12], [13], [3] \rightarrow [6-\sim-21-4-12-13-3]$

Left side of 6 can be 3,10,19. But 3 and 19 are already used. So,

$[6-\sim-21-4-12-13-3] \rightarrow [24-25-\sim-27-22-14-2-23-26-10-6-\sim-21-4-12-13-3]$

And by using last [1],[15],

$$[24-25-11-5-31-18-7-29-20-16-9-27-22-14-2-23-26-10-6-30-19-17-8-28-21-4-12-13-3-1-15]$$

Or

$$[15-1-24-25-11-5-31-18-7-29-20-16-9-27-22-14-2-23-26-10-6-30-19-17-8-28-21-4-12-13-3]$$

But those sequences can't make n be a cyclic number.

Case2) left side of 5 is 4

$[5-31-18-7-29-20-16-9-27-22] \rightarrow [4-5-\sim-22]$

Left side of 4 can be 21 or 12.

(1) Left side of 4 is 21

$[4-5-\sim-22] \rightarrow [6-30-19-17-8-28-21-4-5-\sim-22] \rightarrow [6-\sim-22]$

Side of 15 can be 21,10,1. But 21 is already used. so by using list and previous fact,

$$[1-15-10-26-23]$$

Side of 12 can be 4,13,24. But 4 is already used. so by using list and previous fact,

$$[13-12-24-25-11]$$

left side of 6 can be 10 or 3. But 10 is already used so,

$[6-\sim-22] \rightarrow [3-6-\sim-22]$

Side of 22 can be 14 or 3. But 3 is already used. so, right side of 22 is 14

$[3-6-\sim-22] \rightarrow [3-6-\sim-22-14]$

Left Side of 1 can be 24,15,8,3. But 24,15,8 is already used. so,

$[1-15-10-26-23] \rightarrow [14-22-\sim-6-3-1-15-10-26-23]$

Sequences remain like following;

$[14-22-\sim-6-3-1-15-10-26-23]$, $[13-12-24-25-11]$, $[2]$

It can arrange like following

$$[2-14-22-\sim-6-3-1-15-10-26-23-13-12-24-25-11]$$

or

$$[13-12-24-25-11-14-22-\sim-6-3-1-15-10-26-23-2]$$

But these sequences can't make n be a cyclically number.

(2) Left side of 4 is 12

$[4-5-31-18-7-29-20-16-9-27-22] \rightarrow [12-4-5-\sim-22]$

Left side of 12 can be 24 or 13

1) left side of 12 is 13

$[12-4-5-\sim-22] \rightarrow [13-12-4-5-\sim-22]$

Left side of 13 can be 23 or 3

First, check when left side of 13 is 23

$[13-12-4-5-\sim-22] \rightarrow [23-13-12-4-5-\sim-22]$

By using list,

$[23-13-12-4-5-\sim-22] \rightarrow [10-26-23-13-12-4-5-\sim-22]$

Side of 21 can be 15 or 4. But 4 is already used. so,

$[21-28-8-17-19-30-6] \rightarrow [15-21-\sim-6]$

For the $[24-25-11]$, Side of 24 and 11 can be respectively 12 or 1 and 14 or 5 . But 12 and 5 are already used. so,

$$[24-25-11] \rightarrow [1-24-25-11-14]$$

Sequences remain like following

$[10-26-23-13-12-4-5-\sim-22]$, $[15-21-\sim-6]$, $[1-24-25-11-14]$, $[2]$, $[3]$

It can arrange like following;

$[2-14-11-25-24-1-15-21-\sim-6-10-26-23-13-12-4-5-\sim-22-3]$

or

$[6-\sim-21-15-10-26-23-13-12-4-5-\sim-22-3-1-24-25-11-14-2]$

or

$[10-26-23-13-12-4-5-\sim-22-3-6-\sim-21-15-1-24-25-11-14-2]$

Or

$[15-21-\sim-6-10-26-23-13-12-4-5-\sim-22-3-1-24-25-11-14-2]$

But these sequences can't make n be a cyclically number.

Second, check when left side of 13 is 3

$[13-12-4-5-\sim-22] \rightarrow [3-13-12-4-5-\sim-22]$

Left side of 23 can be 2 or 13 but 13 is already used. so,

$$[23-26-10] \rightarrow [2-23-26-10]$$

Left side of 2 can be 7,14,23. But 7,23 is already used. so,

$$[2-23-26-10] \rightarrow [14-2-23-26-10]$$

Right side of 22 can be 3,14. But only 14 can be possible. So,

$$[3-13-12-4-5-\sim-22] \rightarrow [3-13-12-4-5-\sim-22-14-2-23-26-10]$$

Left side of 11 can be 5 or 14 in the list. But 5 and 14 is already used. so, it's contradiction.

2) left side of 12 is 24

$[12-4-5-\sim-22] \rightarrow [24-12-4-5-\sim-22]$

Left side of 11 can be 5,14. But 5 is already used. by using list and previous fact,

$[24-12-4-5-\sim-22] \rightarrow [14-11-25-24-\sim-22]$

Left side of 14 can be 2,22. But only 2 can be possible and side of 2 can be 7,14,23. But 7 is already used.

By using list and previous fact,

$[14-11-25-24-\sim-22] \rightarrow [10-26-23-2-14-11-25-24-\sim-22] \quad \dots \qquad$ (a)

Side of 13 can be 23,12,3. But 12 is already used. and by using list,

$$[23-13-3] \rightarrow [10-26-23-13-3]$$

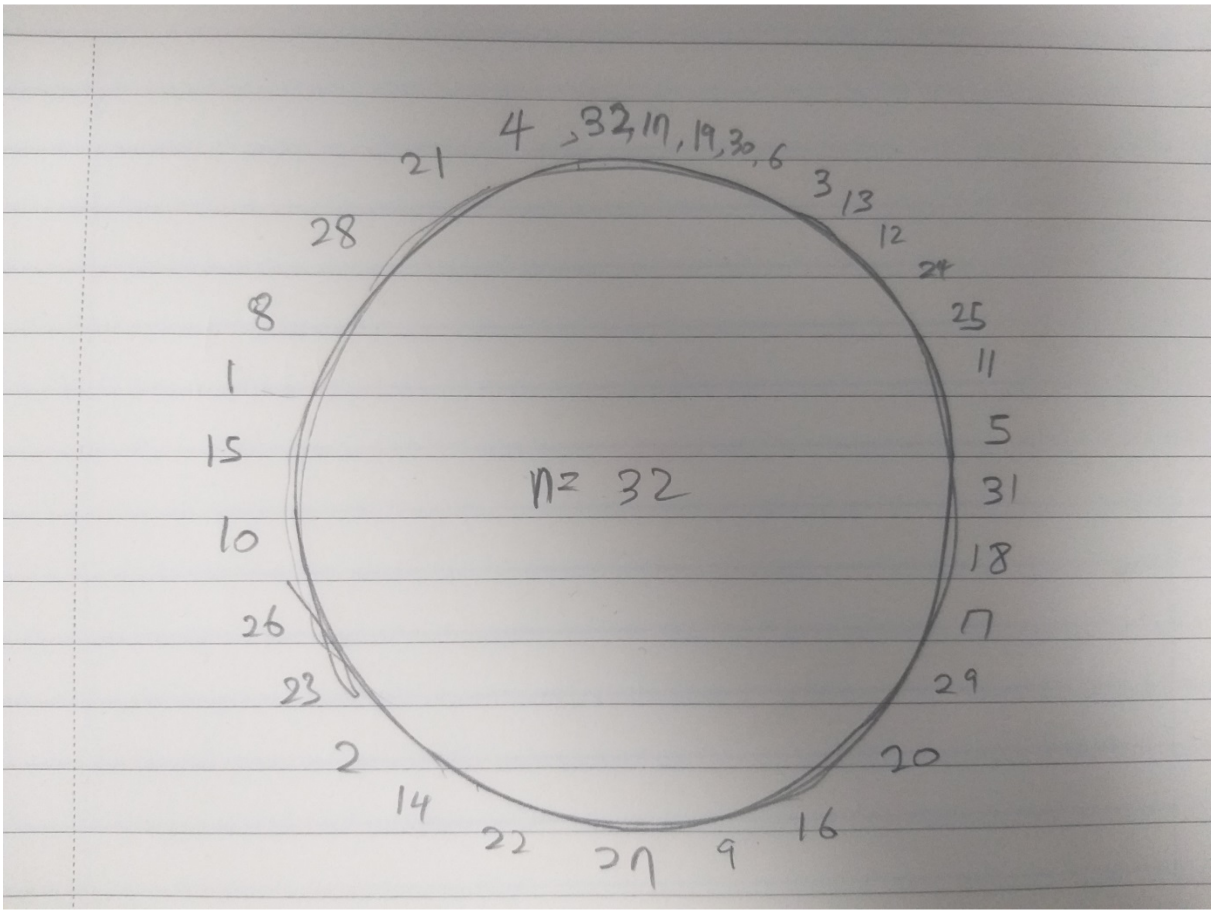
But this sequence is contradicted to sequence (a).

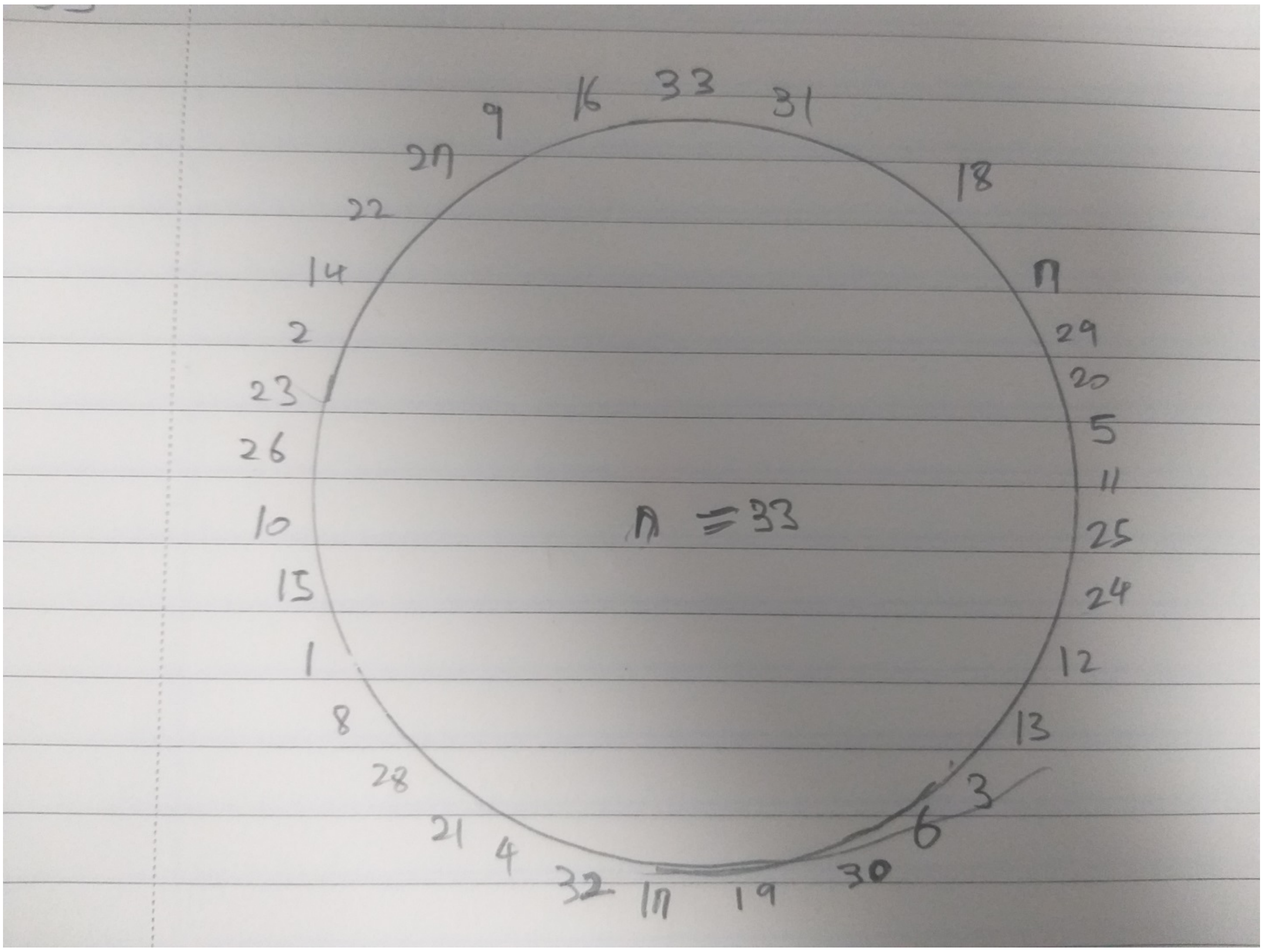
Conclusively, n=31 can't be a cyclic number.

I can find sequences s.t n=32,33 is a cyclically number by using similar way and correcting list.

Ex) (a,i,b)=(4,32,17).

But I don't know how many different sequences that make n=32,33 a cyclically number. The sequences that I find is following;





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