POW2021-19 The answer is zero

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Problem : Suppose that $a_1 + a_2 + \cdots + a_n = 0$ for real numbers $a_1, \ldots, a_n$ and $n \geq 2$. Set $a_{n+i} = a_i$ for $i = 1, 2, \ldots$. Prove that

$$\sum_{i=1}^{n} \frac{1}{a_i(a_i + a_{i+1}) \ldots (a_i + a_{i+1} + \cdots + a_{i+n-2})} = 0$$

Solution : Using induction on n.

For the base case, $n = 2$, $a_2 = -a_1$, and $\frac{1}{a_1} + \frac{1}{a_2} = 0$. So, the statement holds.

Now suppose $n > 2$. Since the denominator is nonzero, any sums of less than $n-1$ consecutive terms are nonzero.

For notational convenience, denote $f_n(a_1, \ldots, a_n) = \sum_{i=1}^{n} \frac{1}{a_i(a_i+a_{i+1})\ldots(a_i+\cdots+a_{i+n-2})}$.

From induction, $f_{n-1}(a_1 + a_2, a_3, a_4, \ldots, a_n) = f_{n-1}(a_1, a_2 + a_3, a_4, \ldots, a_n) = 0$.

I'll show that $a_2 f_n(a_1, \ldots, a_n) + f_{n-1}(a_1 + a_2, \ldots, a_n) = f_{n-1}(a_1, a_2 + a_3, \ldots, a_n)$. Then from induction, $a_2 f_n(a_1, \ldots, a_n) + 0 = 0 \Rightarrow f_n(a_1, \ldots, a_n) = 0$.

Note that

$$f_{n-1}(a_1 + a_2, \ldots, a_n) = \sum_{j=1}^{n} \frac{a_j + a_{j+1} \cdots + a_{n+1}}{a_j(a_j + a_{j+1}) \ldots (a_j + a_{j+1} + \cdots + a_{j+n-2})},$$

$$f_{n-1}(a_1, a_2 + a_3, \ldots, a_n) = \sum_{j=1}^{n} \frac{a_j + a_{j+1} + \cdots + a_{n+2}}{a_j(a_j + a_{j+1}) \ldots (a_j + a_{j+1} + \cdots + a_{j+n-2})}$$

Since each terms of $f_{n-1}(a_1 + a_2, \ldots, a_n)$ is of the form

$$\frac{1}{a_j(a_j + a_{j+1}) \ldots (a_j + \cdots + a_n)(a_j + \cdots + a_n + a_{n+1} + a_{n+2}) \ldots (a_j + \cdots + a_{j+n-2})}$$

for $j = 1, 3, 4, \ldots, n$ and is same as $\frac{a_j+\cdots+a_{n+1}}{a_j(a_j+a_{j+1})\ldots(a_j+\cdots+a_{j+n-2})}$, and when $j = 2$, $\frac{a_2+\cdots+a_{n+1}}{a_2(a_2+a_{2+1})\ldots(a_2+\cdots+a_n)} = 0$.

In case of $f_{n-1}(a_1, a_2 + a_3, \ldots, a_n)$, it can be shown by similar calculation.

$$\begin{aligned}
&a_2 f_n(a_1, \ldots, a_n) + f_{n-1}(a_1 + a_2, \ldots, a_n) \\
&= \sum_{i=1}^{n} \frac{a_2}{a_i(a_i + a_{i+1}) \ldots (a_i + \cdots + a_{i+n-2})} + \sum_{j=1}^{n} \frac{a_j + a_{j+1} + \cdots + a_{n+1}}{a_j(a_j + a_{j+1}) \ldots (a_j + \cdots + a_{j+n-2})} \\
&= \sum_{j=1}^{n} \frac{a_j + \cdots + a_{n+2}}{a_j(a_j + a_{j+1}) \ldots (a_j + \cdots + a_{j+n-2})} \\
&= f_{n-1}(a_1, a_2 + a_3, \ldots, a_n)
\end{aligned}$$

Therefore, $f_n(a_1, \ldots, a_n) = 0$ and proof is done.