POW 2021-18 Independent sets in a tree

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Problem

Let T be a tree (an acyclic connected graph) on the vertex set[n] = {1,…,n}.

Let A be the adjacency matrix of T, i.e., the n x n matrix with Aij = 1 if i and j are adjacent in T and Aij = 0 otherwise. Prove that the number of nonnegative eigenvalues of A equals to the size of the largest independent set of T. Here, an independent set is a set of vertices where no two vertices in the set are adjacent.

Sol

Let T_i be a tree on the vertex set [i] = {1,…,i}, be A_i a its adjacency matrix and O_i be the largest independent set of T_i

Interlacing Theorem(reference)

I can find the theorem in the googling

Lemma 1

Eigenvalue of real Symmetric matrix is real

Proof

Let λ & v be a respectively eigenvalue and eigenvector s.t $\mathrm{Av} = \lambda\mathrm{v}$ and $\mathrm{A} = \mathrm{A}^{\mathrm{T}} = A^{*}$

$$< Av, Av > = (Av)^*Av \ = v^*A^*Av = v^*A^2v = v^*A(\lambda v) = \lambda^2 v^*v = \lambda^2\|v\|^2$$

$\lambda^2 = \frac{<Av,Av>}{\|v\|^2}$ is a nonnegative number. So λ is a real number.

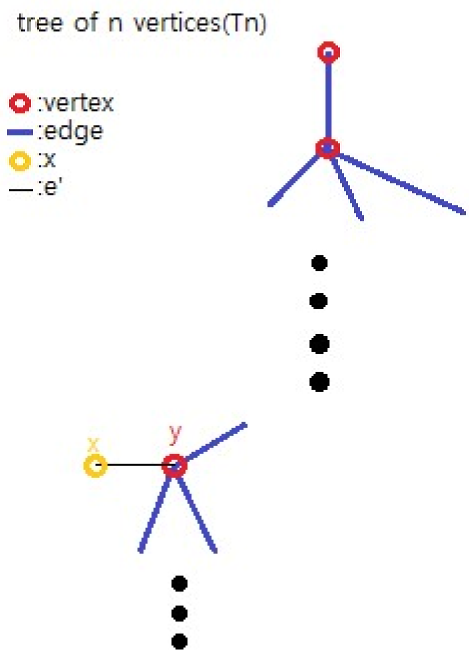
Lemma 2

$|\mathrm{O}_{n+1}| = |\mathrm{O}_n|$ if $y \in \mathrm{O}_n$ for all possible O_n

$|\mathrm{O}_{n+1}| = |\mathrm{O}_n| + 1$ if $y \in \mathrm{O}_n$ for some different O_n

proof

By adding one vertex and one edge from unlabeled T_n, we can make unlabeled T_{n+1}. By removing one vertex and one edge from unlabeled T_{n+1}, we can make unlabeled T_n. It means that all different unlabeled T_{n+1} can be derived from unlabled T_n and vice versa.



<figure 0>

case(1) $y \in \mathrm{O}_n$ for all possible O_n

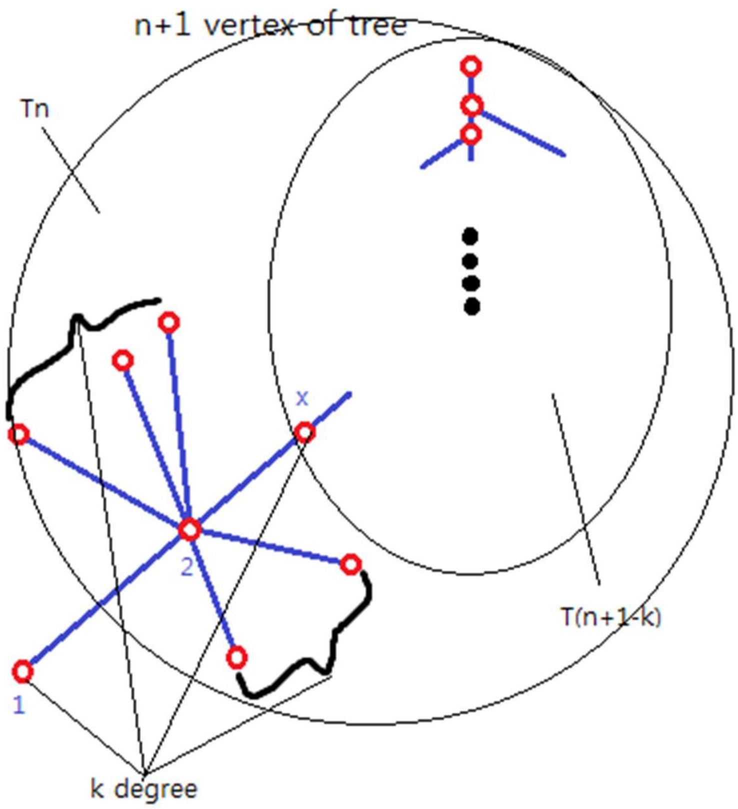
By adding vertex x and adding e', we can make T_{n+1}. If $y \in \mathrm{O}_n$ for all possible different O_n, vertex x can't be a element of some O_{n+1}. If $x \in \mathrm{O}_{n+1}$, y isn't a element of O_{n+1} and $\mathrm{O}_{n+1} - \{x\} = \mathrm{O}_n$. So, it is contradiction that $y \in \mathrm{O}_n$ for all possible different O_n. and because $|\mathrm{O}_{n+1}| \geq |\mathrm{O}_n|$ and x isn't a element of O_{n+1}, $\mathrm{O}_{n+1} \subseteq \mathrm{O}_n$. thus $|\mathrm{O}_{n+1}| = |\mathrm{O}_n|$

case(2) $y \in O_n$ for some different O_n or y isn't always a element of O_n

If $y \in O_n$ for some different O_n or y isn't always a element of O_n, there exists O'_n s.t y is not a element of O'_n and $|O'_n| = |O_n|$. so, $|O'_n + \{x\}| \leq |O_{n+1}|$. And O_{n+1} has always vertex x Because if x can't be a element of O_{n+1} , $O_{n+1} \subseteq O_n$ and it is contradiction. Thus O_{n+1} has always vertex x. Let $|O_{n+1}| \geq . |O'_n| + 2$. Because O_{n+1} has always vertex x, $\exists O_n \ s.t\ O_{n+1} - \{x\} \subseteq O_n$. But $|O_{n+1} - \{x\}| \geq . |O'_n| + 1$ so it is contradiction. Thus $|O_{n+1}| = |O'_n| + 1$

Lemma 3

When $f_n(\lambda) = \det(A_n - \lambda I_n)$, Show that $f_{n+1}(\lambda) = -\lambda * f_n(\lambda) - (-\lambda)^{k-2} * f_{n-k+1}(\lambda)$ with k degree of vertex 2 on the following tree

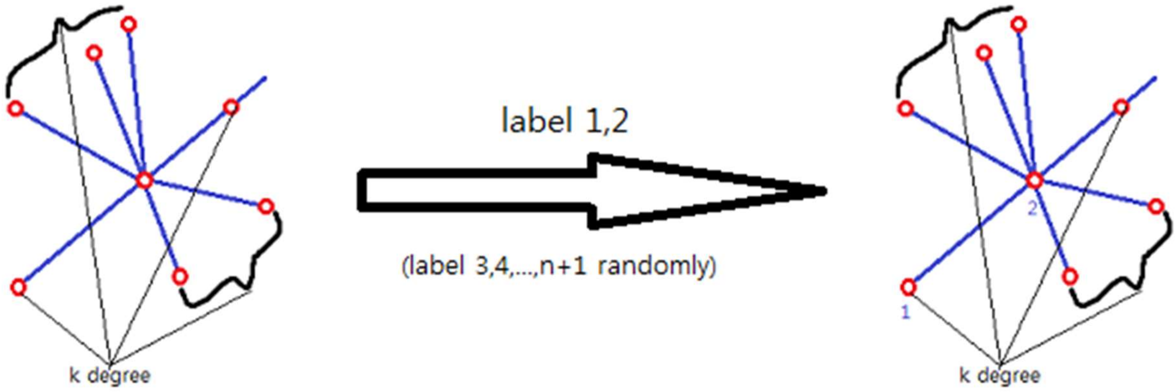


<figure 1>

(I_n is a n x n identity matrix)

Proof

Since n+1 is finite, there always exist like following form on the T_{n+1} and can label 1,2 on the form and k is greater than 2 and less than n, i.e., $\mathrm{n} \geq \mathrm{k} \geq 2$



<figure 2>

$$A_{n+1} - \lambda I_{n+1} = \begin{pmatrix} -\lambda & 1 & \cdots & 0 & 0 \\ 1 & -\lambda & & a_{2,n} & a_{2,n+1} \\ \vdots & & \ddots & \vdots & \\ 0 & a_{n,2} & & -\lambda & \\ 0 & a_{n+1,2} & \cdots & & -\lambda \end{pmatrix} \text{ s.t } \lambda + \sum_{i=1}^{n} a_{i,2} = \mathrm{k},\ a_{i,j} = 0 \text{ for j} = 3, \dots, \mathrm{n}+1$$

$$\left(A_{n+1} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n+1} \\ \vdots & \ddots & \vdots \\ a_{n+1,1} & \cdots & a_{n+1,n+1} \end{pmatrix} \text{ s.t } A_{n+1} = {A_{n+1}}^{T} \text{ and } a_{i,i} = 0 \text{ } for\ i = 1,2, \dots, n+1 \right)$$

$$\mathrm{f}_{\mathrm{n+1}}(\lambda) = \det(\mathrm{A}_{\mathrm{n+1}} - \lambda \mathrm{I}_{\mathrm{n+1}})$$

$$= -\lambda M_{1,1} - M_{(1,1),(2,2)}$$

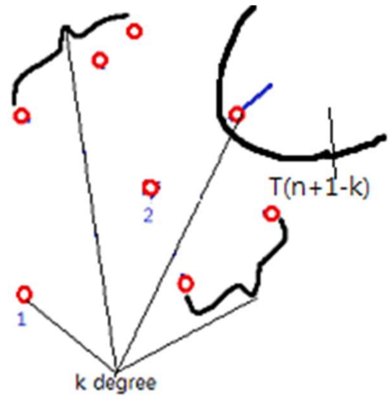
$$= -\lambda * \det \begin{pmatrix} -\lambda & \cdots & a_{2,n+1} \\ \vdots & \ddots & \vdots \\ a_{n+1,2} & \cdots & -\lambda \end{pmatrix} - det \begin{pmatrix} -\lambda & a_{3,4} & \cdots & a_{3,n+1} \\ a_{4,3} & & & \\ \vdots & & \ddots & \vdots \\ a_{n+1,3} & & \cdots & -\lambda \end{pmatrix},$$

$(M_{i,j}\ is\ defined\ to\ be\ the\ determinant\ of\ n\ x\, n\ matrix\ that\ results\ from\ A\ by\ removing\ the\ ith\, row\ and\ jth\ column.$

$and\ M_{(1,1),(2,2)} is\ the\ determinant\ of\ (n-1)\ x\, (n-1)\ matrix\ that\ results\ from\ A_{n+1}$

$by\ removing\ the\ 1st\ row\ \&\ 1st\ column$ and 2nd row & 2nd column)

Graph of $\begin{pmatrix} 0 & a_{3,4} & \cdots & a_{3,n+1} \\ a_{4,3} & 0 & & \\ \vdots & & \ddots & \vdots \\ & & \cdots \quad 0 & \\ a_{n+1,3} & & & 0 \end{pmatrix}$ is like following figure 3.



<figure 3>

And det(its adjacency matrix $-\lambda I_{n-1}$) = $det\begin{pmatrix} -\lambda & a_{3,4} & \cdots & a_{3,n+1} \\ a_{4,3} & & & \\ \vdots & & \ddots & \vdots \\ & & \cdots & \\ a_{n+1,3} & & & -\lambda \end{pmatrix} = \det(G_{n-1})$

$\det(G_{n-1}) = \sum_{j=1}^{n-1}(-1)^{i+j} * [G_{n-1}]_{i,j} * M_{i,j}$ for any i and if we take label number of vertex with 0 degree into i, $\det(G_{n-1}) = det\begin{pmatrix} & & & 0 & & & \\ & & & \vdots & & & \\ & & & 0 & & & \\ 0 & \cdots & 0 & -\lambda & 0 & 0 \ldots & 0 \\ & & & 0 & & & \\ & & & \vdots & & & \\ & & & 0 & & & \end{pmatrix} = -\lambda * M'_{i,i}$

($M'_{i,i}$ is the determinant of (n-2) x (n-2) matrix that results from G_{n-1} by removing ith row and ith column s.t i vertex has 0 degree)

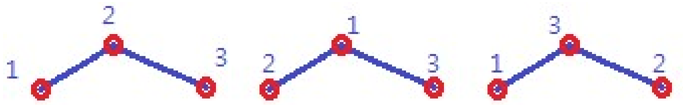
By repeating k-2 times, $\det(G_{n-1}) = (-\lambda)^{k-2} * f_{n-k+1}(\lambda)$ s.t $f_{n-k+1}(\lambda) = \det(A_{n-k+1} - \lambda I_{n-k+1})$ & A_{n-k+1}=adjacency matrix of T_{n-k+1}

Conclusively, $f_{n+1}(\lambda) = -\lambda * f_n(\lambda) - (-\lambda)^{k-2} * f_{n-k+1}(\lambda)$

Lemma 4

Eigenvalues of A_n with labled isomorphic tree are same., Proof

Lemma 4 means that $f_n(\lambda)$ of isomorphic labeled trees are same. For example, $f_n(\lambda)$ of following isomorphic labled trees are same



$f_3(\lambda) = -\lambda(\lambda^2 - 2)$

Assume that f_i for i=1,2,...,n satisfy lemma 4. By lemma 3, $f_{n+1}(\lambda) = -\lambda * f_n(\lambda) - (-\lambda)^{k-2} * f_{n-k+1}(\lambda)$ and $f_n(\lambda), f_{n-k+1}(\lambda)$ satisfy lemma 4. If we randomly arrange from 1 to n+1 into the unlabeled tree T_{n+1}, they are always isomorphic trees and $T_n \& T_{n-k+1}$ are always isomorphic trees. So, $f_n(\lambda)$ and $f_{n-k+1}(\lambda)$ are same for isomorphic trees. Therefore $f_{n+1}(\lambda)$ is always same for isomorphic trees T_{n+1}. $f_1(\lambda) = -\lambda$, $f_2(\lambda) = \lambda^2 - 1$ are always same for isomorphic trees. By mathematical induction Lemma4 is proved

Lemma 5

$f_n(\lambda)$ is expressed like following math form

For n=2t+1 (odd) for t=0,1,2,...

$f_n(\lambda) = -\lambda(\lambda^{2t} + a_{t-1}\lambda^{2t-2} + \cdots + a_2\lambda^4 + a_1\lambda^2 + a_0)$

for n = 2t (even) for t=1,2,3...

$f_n(\lambda) = \lambda^{2t} + a_{t-1}\lambda^{2t-2} + \cdots + a_2\lambda^4 + a_1\lambda^2 + a_0$

(a_i for i = 0,1,...,t-1 are real)

Proof

Assume that f_i for i=1,2,...,n satisfy lemma5

By lemma 4, $f_{n+1}(\lambda) = f_n(\lambda) = -\lambda * f_n(\lambda) - (-\lambda)^{k-2} * f_{n-k+1}(\lambda)$

case(1) n=2t+1

(1)k=2s(even)

$f_{n+1}(\lambda) = f_{2t+2}(\lambda) = -\lambda * f_{2t+1}(\lambda) - (-\lambda)^{k-2} * f_{2t-k+2}(\lambda)$

$= \lambda^2(\lambda^{2t} + a_{t-1}\lambda^{2t-2} + \cdots + a_1\lambda^2 + a_0) - \lambda^{2s-2}(\lambda^{2t-2s+2} + b_{t-s}\lambda^{2t-2s} + \cdots + b_1\lambda^2 + b_0)$

For $s = 1,2$ respectively

$f_{2t+2}(\lambda) = \lambda^{2t+2} + (a_{t-1} - 1)\lambda^{2t} + (a_{t-2} - b_{t-1})\lambda^{2t-2} + \cdots + (a_0 - b_1)\lambda^2 + b_0$ for s=1

$f_{2t+2}(\lambda) = \lambda^{2t+2} + (a_{t-1} - 1)\lambda^{2t} + (a_{t-2} - b_{t-2})\lambda^{2t-2} + \cdots + (a_0 - b_0)\lambda^2$ for s=2

For $s \geq 3$

$f_{2t+2}(\lambda) = \lambda^{2t+2} + (a_{t-1} - 1)\lambda^{2t} + (a_{t-2} - b_{t-s})\lambda^{2t-2} + \cdots + (a_{s-2} - b_0)\lambda^{2s-2} + \cdots + a_0\lambda^2$

(2)k=2s+1(odd)

$f_{n+1}(\lambda) = f_{2t+2}(\lambda) = -\lambda * f_{2t+1}(\lambda) - (-\lambda)^{k-2} * f_{2t-k+}\ (\lambda)$

$= \lambda^2(\lambda^{2t} + a_{t-1}\lambda^{2t-2} + \cdots + a_1\lambda^2 + a_0) + \lambda^{2s-1} * -\lambda(\lambda^{2t-2s} + b_{t-s-1}\lambda^{2t-2s} + \cdots + b_1\lambda^2 + b_0)$

$= \lambda^{2t+2} + (a_{t-1} - 1)\lambda^{2t} + (a_{t-2} - b_{t-s-1})\lambda^{2t-2} + \cdots + (a_{s-1} - b_0)\lambda^{2s} + \cdots + a_0\lambda^2$ for $s \geq 1$

So, $f_{2t+2}(\lambda)$ is expressed like $\lambda^{2t+2} + c_t\lambda^{2t} + \cdots + c_2\lambda^4 + c_1\lambda^2 + c_0$ for $n \geq k \geq 2$

case(2) n=2t

(1)k=2s(even)

$f_{n+1}(\lambda) = f_{2t+1}(\lambda) = -\lambda * f_{2t}(\lambda) - (-\lambda)^{k-2} * f_{2t-k+1}(\lambda)$

$= -\lambda(\lambda^{2t} + a_{t-1}\lambda^{2t-2} + \cdots + a_1\lambda^2 + a_0) - \lambda^{2s-2} * -\lambda(\lambda^{2t-2} + b_{t-s-1}\lambda^{2t-2s-2} + \cdots + b_1\lambda^2 + b_0)$

$= -\lambda(\lambda^{2t} + (a_{t-1} - 1)\lambda^{2t-2} + (a_{t-2} - b_{t-s-1})\lambda^{2t-4} + \cdots + (a_{s-1} - b_0)\lambda^{2s-2} + \cdots + a_0)\ for\ s \geq 1$

(2)k=2s+1(odd)

$f_{n+1}(\lambda) = f_{2t+1}(\lambda) = -\lambda * f_{2t}(\lambda) - (-\lambda)^{k-2} * f_{2t-k+}\ (\lambda)$

$= -\lambda(\lambda^{2t} + a_{t-1}\lambda^{2t-2} + \cdots + a_1\lambda^2 + a_0) + \lambda^{2s-1} * (\lambda^{2t-2s} + b_{t-s-1}\lambda^{2t-2s-2} + \cdots + b_1\lambda^2 + b_0)$

$= -\lambda(\lambda^{2t} + (a_{t-1} - 1)\lambda^{2t-2} + (a_{t-2} - b_{t-s-1})\lambda^{2t-4} + \cdots + (a_{s-1} - b_0)\lambda^{2s-2} + \cdots + a_0)\ for\ s \geq 1$

So, $f_{2t+1}(\lambda)$ is expressed like $-\lambda(\lambda^{2t} + c_{t-1}\lambda^{2t-2} + \cdots + c_2\lambda^4 + c_1\lambda^2 + c_0)$ for $n \geq k \geq 2$

Conclusively, when f_i for i=1,2,…,n satisfy lemma5, f_{n+1} satisfy lemma5 and

for n=1,

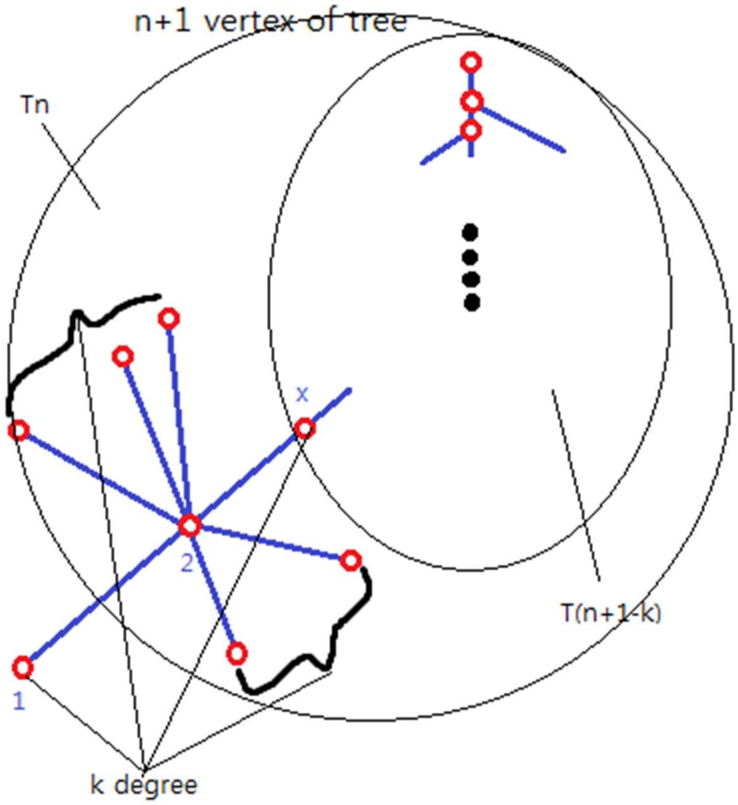
$A_1 = (0) \rightarrow f_1(\lambda) = \det(A_1 - \lambda I_1) = -\lambda$

For n=2,

$A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow f_2(\lambda) = \det(A_2 - \lambda I_2) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1$

By mathematical induction, lemma 5 is proved.

I will derive the number of nonnegative eigenvalues of A_{n+1} equals to the size of O_{n+1} by mathematical induction. Assume that the number of nonnegative eigenvalues of A_i for i=1,2,…,n equals to the size of O_n. We can make the labeled T_{n+1} by adding vertex (n+1) and edge(n+1,p) to T_n and By lemma 4, change the vertex n+1,p respectively to the vertex 1,2. It means that we can make labeled T_{n+1} of <figure 1> by adding vertex 1 and edge(1,2) to T_n. By lemma 3, $f_{n+1}(\lambda) = -\lambda * f_n(\lambda) - (-\lambda)^{k-2} * f_{n-k+1}(\lambda)$ with figure 1



<figure 1>

For $k \geq 3$, vertex 2 is a element of some different O_n or is not always a element of O_n. Let vertex 2 $\in \mathrm{O_n}$ for all possible $\mathrm{O_n}$ but there exists $\mathrm{O'_n} = \mathrm{O_n} - \{2\} + \sum\{v\}$ s.t v is adjacent to 2($\mathrm{v} \neq$ vertex 1) and $|\mathrm{O'_n}| \geq |\mathrm{O_n}|$. It is contradiction. Thus vertex 2 is case(2) of Lemma 2.

Therefore,

$$|\mathrm{O_{n+1}}| = |\mathrm{O_n}| + 1 \;\&\; |\mathrm{O_{n-k+1}}| = |\mathrm{O_n}| - (k-2) \; for \; k \geq 3 \qquad \dots \qquad \text{(a)\&(b)}$$

Proof of (a)&(b)

because vertex 2 is case(2) of lemma 2, (a) is proved. when vertex x is case(1) of Lemma 2 with $\mathrm{T_{n-k+1}}$, $\mathrm{O_{n-k+1}} + \sum\{v\} = \mathrm{O_n}$ s.t v is adjacent to 2($\mathrm{v} \neq$ vertex 1) because $\mathrm{O_{n-k+1}} + \sum\{v\}$ is the biggest size of possible independent set. when vertex x is case(2) of Lemma 2 with $\mathrm{T_{n-k+1}}$, also $\mathrm{O_{n-k+1}} + \sum\{v\} = \mathrm{O_n}$ by same above reason. So, whether vertex x is case(1) or case(2) of Lemma 2,

$|\mathrm{O_{n-k+1}}| = |\mathrm{O_n}| - |\sum\{\mathrm{v}\}| = |\mathrm{O_n}| - (k-2)$. (b) is proved

For k=2, vertex 2 can be case(1) or case(2) of Lemma 2.

(1) x is case(1) of lemma 2 ($|O_n| = |O_{n-k+1}| = |O_{n-1}|$)

Obviously, vertex 2 is case(2) of lemma 2. So, $|O_{n+1}| = |O_n| + 1$

(2) x is case(2) of lemma 2 ($|O_n| = |O_{n-k+1}| + 1 = |O_{n-1}| + 1$)

Because x is case(2) of lemma2, vertex 2 is always a element of O_n. So, vertex 2 is case(1) of lemma 2 and $|O_{n+1}| = |O_n|$

Conclusively, we can know that whether x is case(1) or not case(2) of lemma 2,

$$|O_{n+1}| = |O_{n-1}| + 1$$

case(1) n=2t

By lemma 1, $\mathrm{f_{2t}(\lambda)}$ has 2t real eigenvalues and By lemma 5, $\mathrm{f_{2t}(\lambda) = f_{2t}(-\lambda)}$. So we can order its eigenvalues like this

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 0 \geq -\lambda_t \geq \cdots \geq -\lambda_2 \geq -\lambda_1$$

$$(\mathrm{f_{2t}(\lambda)} = \lambda^{2t} + a_{t-1}\lambda^{2t-2} + \cdots + a_1\lambda^2 + a_0)$$

Let $\lambda_{t-q+1} = \dots = \lambda_{t-1} = \lambda_t = 0$. (# of zero eigenvalues = 2q).

$$\text{Then, } f_{2t}(\lambda) = \lambda^{2t} + \cdots + a_0 = \lambda^{2q}\left(\lambda^{2t-2q} + a_{t-1}\lambda^{2t-2-2q} + \cdots + a_q\right) \text{ s.t } a_q \neq 0$$

(1)k=2s for s=2,3,...

By using similar way, eigenvalues of $f_{2t-2}\quad(\lambda)$ can be ordered like this

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{t-s} \geq \lambda_0 = 0 \geq -\lambda_{t-s} \geq \cdots \geq -\lambda_2 \geq -\lambda_1$$
$$(t-s \geq 1, f_{2t-2s}\quad(\lambda) = -\lambda(\lambda^{2t-2} + b_{t-s-1}\lambda^{2t-2s-2} + \cdots + b_0))$$

Let $\lambda_{t-s-q'+1} = \dots = \lambda_{t-s-1} = \lambda_{t-s} = 0$. (# of zero eigenvalues = $2q' + 1$)

$$\text{Then, } f_{2t-2s}\quad(\lambda) = -\lambda^{2q'+1}(\lambda^{2t-2s-2q'} + b_{t-s-1}\lambda^{2t-2s-2-2q'} + \cdots + b_{q'}) \text{ s.t } b_{q'} \neq 0$$

By assumption, $|O_{2t}|$= # of nonnegative eigenvalues = $t+q$, $|O_{2t-2s+1}| = t-s+q'+1$

$|O_{n-k+1}| = |O_n| - (k-2) \rightarrow |O_{2t-2s}\quad| = |O_{2t}| - (2s-2) \rightarrow t-s+q'+1 = t+q-(2s-2)$

$\rightarrow q' = q-s+1$

Thus, $f_{2t+1}(\lambda) = -\lambda * f_{2t}(\lambda) - (-\lambda)^{2s-2} * f_{2t-2s+1}(\lambda)$

$= -\lambda^{2q+1}\left(\lambda^{2t-2q} + a_{t-1}\lambda^{2t-2-2q} + \cdots + a_q\right) + \lambda^{2s-2}\lambda^{2q'+1}(\lambda^{2t-2s-2q'} + b_{t-s-1}\lambda^{2t-2s-2-2q'} + \cdots + b_{q'})$

$= -\lambda^{2q+1}\left(\lambda^{2t-2q} + a_{t-1}\lambda^{2t-2-2q} + \cdots + a_q\right) + \lambda^{2q+1}(\lambda^{2t-2-} \quad + b_{t-s-1}\lambda^{2t-4-2q} + \cdots + b_{q-s+1})$

$= -\lambda^{2q+1}(\lambda^{2t-2q} + (a_{t-1} - 1)\lambda^{2t-2-2q} + (a_{t-2} - b_{t-s-1})\lambda^{2t-4-} \quad + \cdots + (a_q - b_{q-s+1}))$

(2)k=2s+1 for s=1,2,...

eigenvalues of $f_{2t-2s}(\lambda)$ can be ordered like this,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{t-s} \geq 0 \geq -\lambda_{t-s} \geq \cdots \geq -\lambda_2 \geq -\lambda_1$$
$$(f_{2t-2s}(\lambda) = \lambda^{2t-2s} + b_{t-s-1}\lambda^{2t-2s-2} + \cdots + b_0)$$

Let $\lambda_{t-s-q'+1} = \dots = \lambda_{t-s-1} = \lambda_{t-s} = 0$(# of zero eigenvalues = $2q'$)

$$\text{Then, } f_{2t-2s}(\lambda) = \lambda^{2q'}\left(\lambda^{2t-2s-2q'} + b_{t-s-1}\lambda^{2t-2s-2-2q'} + \cdots + b_{q'}\right) \text{ s.t } b_{q'} \neq 0$$

By assumption, $|O_{2t}|=$ # of nonnegative eigenvalues $= \mathrm{t} + \mathrm{q}$, $|O_{2t-2s}| = t - s + q'$

$|\mathrm{O}_{\mathrm{n-k+1}}| = |\mathrm{O}_{\mathrm{n}}| - (k-2) \;\rightarrow\; |\mathrm{O}_{\mathrm{2t-2s}}| = |\mathrm{O}_{\mathrm{2t}}| - (2s-1) \;\rightarrow\; t - s + q' \;=\; \mathrm{t} + \mathrm{q} - (2\mathrm{s} - 1)$

$\rightarrow\; q' \;=\; \mathrm{q} - \mathrm{s} + 1$

Thus, $\mathrm{f}_{2t+1}(\lambda) = -\lambda * \mathrm{f}_{2t}(\lambda) - (-\lambda)^{2s-1} * \mathrm{f}_{2t-2}\ (\lambda)$

$= -\lambda^{2q+1}\left(\lambda^{2t-2q} + \mathrm{a}_{t-1}\lambda^{2t-2-2q} + \cdots + \mathrm{a}_{q}\right) + \lambda^{2s-1}\lambda^{2q'}(\lambda^{2t-2s-2q'} + \ b_{t-s-1}\lambda^{2t-2s-2-2q'} + \cdots + \mathrm{b}_{q'})$

$= -\lambda^{2q+1}\left(\lambda^{2t-2q} + \mathrm{a}_{t-1}\lambda^{2t-2-} \quad + \cdots + \mathrm{a}_{q}\right) + \lambda^{2q+1}(\lambda^{2t-2-2q} + \ b_{t-s-1}\lambda^{2t-4-} \quad + \cdots + \mathrm{b}_{q-s+1})$

$= -\lambda^{2q+1}\left(\lambda^{2t-2} \ + \mathrm{a}_{t-1}\lambda^{2t-2-2q} + \cdots + \mathrm{a}_{q}\right) + \lambda^{2q+1}(\lambda^{2t-2-} \quad + \ b_{t-s-1}\lambda^{2t-4-} \quad + \cdots + \mathrm{b}_{q-s+1})$

$= -\lambda^{2q+1}(\lambda^{2t-2} \ + (\mathrm{a}_{t-1} - 1)\lambda^{2t-2-2q} + (\mathrm{a}_{t-2} - b_{t-s-1})\lambda^{2t-4-2q} + \cdots + (\mathrm{a}_{q} - \mathrm{b}_{q-s+1}))$

Therefore, for $3 \leq \mathrm{k} \leq \mathrm{n}$, $\mathrm{f}_{2t+1}(\lambda)$ has at least 2q+1 zero eigenvalues.

And By interlacing theorem,

$$\lambda_1' \geq \lambda_1 \geq \lambda_2' \geq \cdots \geq \lambda_{t-q+1}' \geq \lambda_{t-q+1} \geq \lambda_{t-q+2}' \geq \cdots \geq \lambda_t' \geq \lambda_t \geq \ \lambda_0' = 0 \geq -\lambda_t' \geq \cdots \geq -\lambda_1'$$

(s.t $\mathrm{f}_{2t+1}(\pm\lambda_i') = 0$ for i=0,1,...,t and $\mathrm{f}_{2t}(\pm\lambda_i) = 0$ for i=1,2,...t & $\lambda_{t-q+1} = \cdots = \lambda_t = 0$)

If $\lambda_{t-q}' = 0$, $\lambda_{t-q}' \geq \lambda_{t-q} \geq \lambda_{t-q+1}' \rightarrow \lambda_{t-q} = 0$. Because $\lambda_{t-q} > \lambda_{t-q+1} = 0$, It is contradiction. So, $\lambda_{t-q+1}' = 0 \;\rightarrow$, $\mathrm{f}_{2t+1}(\lambda)$ has t+q+1 nonnegative eigenvalues. And By (a), $|\mathrm{O}_{2t+1}| = |\mathrm{O}_{2t}| + 1 = \mathrm{t} + \mathrm{q} + 1$

Conclusively, $|\mathrm{O}_{2t+1}| = \#\text{ of nonnegative eigenvalues of }\mathrm{A}_{2t+1}\text{ for }3 \leq \mathrm{k} \leq \mathrm{n} = 2\mathrm{t}$

(3)k=2

(1) x is case(1) of lemma 2 ($|O_n| = |O_{n-k+1}| = |O_{n-1}|$)

Because vertex 2 is case(2) of lemma 2, it is proved by above k=2s for s=1

(2) x is case(2) of lemma 2 ($|O_n| = |O_{n-k+1}| + 1 = |O_{n-1}| + 1$)

By using similar way,

$$\mathrm{f}_{n+1}(\lambda) = -\lambda * \mathrm{f}_{n}(\lambda) - (-\lambda)^{k-2} * \mathrm{f}_{n-k+1}(\lambda) \quad \rightarrow \quad \mathrm{f}_{2t+1} = -\lambda * \mathrm{f}_{2t}(\lambda) - \mathrm{f}_{2t-1}(\lambda)$$

$f_{2t}(\lambda) = \lambda^{2q}(\lambda^{2t-} \quad + a_{t-1}\lambda^{2t-2-2q} + \cdots + a_q)$ $s.t\, a_q \neq 0$ (# of zero eigenvalues = $2q$)

$f_{2t-1}(\lambda) = -\lambda^{2q'+1}(\lambda^{2t-2-2\ '} + b_{t-2}\lambda^{2t-4-2\ '} + \cdots + b_{q'})$ $s.t\ b_{q'} \neq 0$ (# of zero eigenvalues = $2q' + 1$)

By assumption, $|O_{2t}| = t + q$, $|O_{2t-1}| = t + q'$ and x is case(2) of lemma 2

$|O_n| = |O_{n-1}| + 1 \rightarrow |O_{2t}| = |O_{2t-1}| + 1 \rightarrow t + q = t + q' + 1$

$\rightarrow q' = q - 1$

Thus, $f_{2t+1} = -\lambda * f_{2t}(\lambda) - f_{2t-1}(\lambda)$

$= -\lambda^{2q+1}(\lambda^{2t-2q} + a_{t-1}\lambda^{2t-2-} \quad + \cdots + a_q) + \lambda^{2q'+1}(\lambda^{2t-2-2\ '} + b_{t-2}\lambda^{2t-4-2\ '} + \cdots + b_{q'})$

$= -\lambda^{2q+1}(\lambda^{2t-2q} + a_{t-1}\lambda^{2t-2-2q} + \cdots + a_q) + \lambda^{2q-1}(\lambda^{2t-2q} + b_{t-2}\lambda^{2t-2-2q} + \cdots + b_{q-1})$

$= -\lambda^{2q-1}(\lambda^{2t-2q} \quad + (a_{t-1} - 1)\lambda^{2t-2q} + (a_{t-2} - b_{t-2})\lambda^{2t-2-} \quad \ldots + (a_q - b_q)\lambda^2 - b_{q-1})$

therefore $b_{q-1} \neq 0$ and $f_{2t+1}(\lambda)$ has 2q-1 zero eigenvalues for $k = 2$.

Thus, $f_{2t+1}(\lambda)$ has t+q nonnegative eigenvalues and since vertex 2 is case(1) of lemma 2,

$|O_{n+1}| = |O_n| \rightarrow |O_{2t+1}| = |O_{2t}| = t + q$

Conclusively, $|O_{2t+1}| =$ # of nonnegative eigenvalues of A_{2t+1} for $2 \leq k \leq n = 2t$

I will show case(2) n=2t+1 by using same method. It's just simple calculation for proof

case(2) n=2t+1

$f_{2t+1}(\lambda)$ can be ordered like following,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq \lambda_0 = 0 \geq -\lambda_t \geq \cdots \geq -\lambda_2 \geq -\lambda_1$$
$$(f_{2t+1}(\lambda) = -\lambda(\lambda^{2t} + a_{t-1}\lambda^{2t-2} + \cdots + a_1\lambda^2 + a_0))$$

Let $\lambda_{t-q+1} = \ldots = \lambda_{t-1} = \lambda_t = 0$. (# of zero eigenvalues = $2q + 1$).

$$\text{Then, } f_{2t+1}(\lambda) = -\lambda^{2q+1}(\lambda^{2t-2} \quad + a_{t-1}\lambda^{2t-2-2q} + \cdots + a_q) \text{ s.t } a_q \neq 0$$

(1)k=2s for s=2,3,…

eigenvalues of $f_{2t-2s+2}(\lambda)$ can be ordered like this

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{t-s+1} \geq 0 \geq -\lambda_{t-s+1} \geq \cdots \geq -\lambda_2 \geq -\lambda_1$$

$$(f_{2t-2s}\quad(\lambda) = \lambda^{2t-2s}\quad + b_{t-s}\lambda^{2t-2s} + \cdots + b_0)$$

Let $\lambda_{t-s-q'+2} = \ldots = \lambda_{t-s-1} = \lambda_{t-s+1} = 0$. (# of zero eigenvalues = 2q')

$$\text{Then, } f_{2t-2s}\quad(\lambda) = \lambda^{2q'}(\lambda^{2t-2s+2-2q'} + b_{t-s}\lambda^{2t-2s-2q'} + \cdots + b_{q'}) \text{ s.t } b_{q'} \neq 0$$

By assumption, $|O_{2t+1}| = t+q+1$, $|O_{2t-2s+2}| = t-s+1+q'$ and $|O_{n-k+1}| = |O_n| - (k-2)$

$\rightarrow q' = q-s+2$

Thus, $f_{2t+2}(\lambda) = -\lambda * f_{2t+1}(\lambda) - (-\lambda)^{2s-2} * f_{2t-2s+2}(\lambda)$

$= \lambda^{2q+2}(\lambda^{2t-2q} + (a_{t-1}-1)\lambda^{2t-2-2q} + (a_{t-2} - b_{t-s})\lambda^{2t-4-}\quad + \cdots + (a_q - b_{q-s+2}))$

(2)k=2s+1 for s=1,2,...

eigenvalues of $f_{2t-2s+1}(\lambda)$ can be ordered like this, (# of zero eigenvalues = $2q'+1$)

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{t-s} \geq \lambda_0 = 0 \geq -\lambda_{t-s} \geq \cdots \geq -\lambda_2 \geq -\lambda_1$$

$$(t-s \geq 1, f_{2t-2s+1}(\lambda) = -\lambda(\lambda^{2t-2} + b_{t-s-1}\lambda^{2t-2s}\quad + \cdots + b_0))$$

Let $\lambda_{t-s-q'+1} = \ldots = \lambda_{t-s-1} = \lambda_{t-s} = 0$

$$\text{Then, } f_{2t-2s+1}(\lambda) = -\lambda^{2q'+1}(\lambda^{2t-2s-2q'} + b_{t-s-1}\lambda^{2t-2s-2-2q'} + \cdots + b_{q'}) \text{ s.t } b_{q'} \neq 0$$

By assumption, $|O_{2t+1}| = t+q+1$, $|O_{2t-2s+1}| = t-s+1+q'$ and $|O_{n-k+1}| = |O_n| - (k-2)$

$\rightarrow q' = q-s+2$

Thus, $f_{2t+2}(\lambda) = -\lambda * f_{2t+1}(\lambda) - (-\lambda)^{2s-1} * f_{2t-2s}\quad(\lambda)$

$= \lambda^{2q+2}(\lambda^{2t-2q} + (a_{t-1}-1)\lambda^{2t-2-2q} + (a_{t-2} - b_{t-s})\lambda^{2t-4-2q} + \cdots + (a_q - b_{q-s+2}))$

Therefore, for $3 \leq k \leq n$, $f_{2t+2}(\lambda)$ has at least 2q+2 zero eigenvalues.

And By interlacing theorem,

$$\lambda'_1 \geq \lambda_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{t-q+1} \geq \lambda_{t-q+1} \geq \lambda'_{t-q+2} \geq \cdots \geq \lambda_t \geq \lambda'_{t+1} \geq \lambda_0 = 0 \geq -\lambda'_{t+1} \geq \cdots \geq -\lambda'_1$$

(s.t $f_{2t+2}(\pm\lambda_i') = 0$ for i=1,...,t,t+1 and $f_{2t+1}(\pm\lambda_i) = 0$ for i=0,1,2,...t & $\lambda_{t-q+1} = \cdots = \lambda_t = 0$)

If $\lambda_{t-q}' = 0$, $\lambda_{t-q}' \geq \lambda_{t-q} \geq \lambda_{t-q+1}' \rightarrow \lambda_{t-q} = 0$. Because $\lambda_{t-q} > \lambda_{t-q+1} = 0$, It is contradiction. So, $\lambda_{t-q+1}' = 0 \rightarrow f_{2t+2}(\lambda)$ has t+q+2 nonnegative eigenvalues. And By (a), $|O_{2t+2}| = |O_{2t+1}| + 1 = t + q + 2$

Conclusively, $|O_{2t+2}| = \#\text{ of nonnegative eigenvalues of } A_{2t+1} \text{ for } 3 \leq k \leq n$

(3)k=2

(1) x is case(1) of lemma 2 ($|O_n| = |O_{n-k+1}| = |O_{n-1}|$)

Because vertex 2 is case(2) of lemma 2, it is proved by above k=2s for s=1

(2) x is case(2) of lemma 2 ($|O_n| = |O_{n-k+1}| + 1 = |O_{n-1}| + 1$)

By using similar way,

$$f_{n+1}(\lambda) = -\lambda * f_n(\lambda) - (-\lambda)^{k-2} * f_{n-k+1}(\lambda) \quad \rightarrow \quad f_{2t+2} = -\lambda * f_{2t+1}(\lambda) - f_{2t}(\lambda)$$

$f_{2t+1}(\lambda) = -\lambda^{2q+1}(\lambda^{2t-2q} + a_{t-1}\lambda^{2t-2-} + \cdots + a_q)$ s.t $a_q \neq 0$ (# of zero eigenvalues = $2q+1$)

$f_{2t}(\lambda) = -\lambda^{2q'}(\lambda^{2t-2q'} + b_{t-1}\lambda^{2t-2-2q'} + \cdots + b_{q'})$ s.t $b_{q'} \neq 0$ (# of zero eigenvalues = $2q'$)

By assumption, $|O_{2t+1}| = t + q + 1$, $|O_{2t}| = t + q'$ and x is case(2) of lemma 2

$|O_n| = |O_{n-1}| + 1 \rightarrow |O_{2t+1}| = |O_{2t}| + 1 \rightarrow t + q + 1 = t + q' + 1$

$\rightarrow q' = q$

Thus, $f_{2t+2} = -\lambda * f_{2t+1}(\lambda) - f_{2t}(\lambda)$

$= \lambda^{2q+2}(\lambda^{2t-2q} + a_{t-1}\lambda^{2t-2-2q} + \cdots + a_q) - \lambda^{2q'}(\lambda^{2t-2q'} + b_{t-1}\lambda^{2t-2-2\ '} + \cdots + b_{q'})$

$= \lambda^{2q+2}(\lambda^{2t-2q} + a_{t-1}\lambda^{2t-2-2q} + \cdots + a_q) - \lambda^{2q}(\lambda^{2t-2} + b_{t-1}\lambda^{2t-2-2q} + \cdots + b_q)$

$= \lambda^{2q}(\lambda^{2t+2-2q} + (a_{t-1} - 1)\lambda^{2t-2} + (a_{t-2} - b_{t-1})\lambda^{2t-2-2q} + \cdots + (a_q - b_{q+1})\lambda^2 - b_q)$

therefore $b_q \neq 0$ and $f_{2t+1}(\lambda)$ has 2q zero eigenvalues for $k = 2$.

Thus, $f_{2t+1}(\lambda)$ has t+1+q nonnegative eigenvalues and since vertex 2 is case(1) of lemma 2,

$|O_{n+1}| = |O_n| \rightarrow |O_{2t+2}| = |O_{2t+1}| = t + q + 1$

Conclusively, $|O_{2t+2}|=$ # of nonnegative eigenvalues of A_{2t+2} for $2 \leq k \leq n = 2t+1$

For n=1,2,3, its $f_n(\lambda) = -\lambda,\ \lambda^2 - 1,\ -\lambda(\lambda^2 - 2)$ and $|O_n| = 1, 1, 2$ respectively. It can show easily. So by mathematical induction, the number of nonnegative eigenvalues of A_n equals to the size of the largest independent set of T_n