POW 2021-16 Optimal constant

전해구(기계공학과 졸업생)

September 27, 2021

Problem

For a given positive integer n and a real number a, find the maximum constant b such that

$$x1^n + x2^n + \cdots + xn^n + ax1x2 \ldots xn \geq b(x1 + x2 + \cdots + xn)^n$$

for any non-negative $x1, x2, \ldots, xn$

sol

problem of equation can be expressed like following;

$$\frac{x1^n + x2^n + \cdots + xn^n + ax1x2 \ldots xn}{(x1 + x2 + \cdots + xn)^n} \geq b$$

For $x1 + x2 + \cdots + xn = T > 0$ and $f(x) = x1^n + x2^n + \cdots + xn^n + ax1x2 \ldots xn$,

max(b) = $\min\left(\frac{f(x)}{T^n}\right)$.

Therefore max(b) = min (local extremum, boundary value) for $\frac{f(x)}{T^n}$

1)minimum of Boundary value

Boundary value means value of boundary conditions

Boundary conditions means that $(x1, x2, \ldots, xn)$ such that there exists $xi = 0$ for $i = 1,2, \ldots, n$

ex) $\left(\frac{T}{n-1}, \frac{T}{n-1}, \dots, \frac{T}{n-1}, 0\right) or$ for n=4, $\left(\frac{T}{2}, \frac{T}{4}, \frac{T}{4}, 0\right)$

so boundary conditions expressed like following;

$(x1, x2, \dots, xn) = (y1, y2, \dots, yn-1, 0)$ such that $y1 + y2 + \cdots + yn - 1 = T$

So, $f(x)\ /T^n = y1^n + y2^n + \cdots + yn - 1^n$

by AM-GM(arithmetic Mean-Geometric Mean inequality)

$\therefore \min\left(\frac{f(x)}{T^n}\right) = \frac{1}{(n-1)^{n-1}}$ such that $y1 = y2 \dots = yn = \frac{T}{n-1}$

2)minimum of local extremum (except boundary condition)

By lagrange multiplier method, local extreme points satisfy following condition;

For $f(x) = x1^n + x2^n + \cdots + xn^n + ax1x2 \dots xn,\ g(x) = x1 + x2 + \cdots + xn - T = 0,$

There exists λ such that $\nabla f = \lambda * \nabla g$

($\nabla \mathbf{f} = \left(\frac{\delta f}{\delta x1}, \frac{\delta f}{\delta x2}, \dots, \frac{\delta f}{\delta xn}\right)$ such that $\frac{\delta f}{\delta xi} = \mathbf{nxi^{n-1} + ax1 *..(xi-1)*(xi+1)*..* xn},\ \nabla \mathbf{g} = (1,1,\dots,1)$)

Assume that different x1, x2, x3 exist such that $\nabla \mathbf{f} = \lambda * \nabla \mathbf{g}$ $(x1 \neq x2, x1 \neq x3, x2 \neq x3)$

$\mathrm{nx1^{n-1} + ax2x3 \dots xn = \lambda}$ … (1)

$\mathrm{nx2^{n-1} + ax1x3 \dots xn = \lambda}$ … (2)

$\mathrm{nx3^{n-1} + ax1x2 \dots xn = \lambda}$ … (3)

(1) *x1=(2)*x2 $\Rightarrow$ $\mathrm{n(x1^{n-1} + x1^{n-2}x2 \dots + x1x2^{n-2} + x2^{n-1}) = \lambda}$ … (a)

(1) *x2=(3)*x3 $\Rightarrow$ $\mathrm{n(x1^{n-1} + x1^{n-2}x3 \dots + x1x3^{n-2} + x3^{n-1}) = \lambda}$ … (b)

(a)=(b) $\Rightarrow$ $x1^{n-2} + x1^{n-3}(x2 + x3) \ldots + (x2^{n-2} + x2^{n-1}x3 + \cdots + x3^{n-2)} = 0$

$\Rightarrow$ $x1 = x2 = x3 = 0$ $\Rightarrow$ boundary condition !!

Excepting boundary condition, there not exist three different x1,x2,x3 and at most exist two different x1,x2.

For only one x1 exist such that $x1 = x2 = \cdots = xn = \frac{T}{n}$, $\therefore \frac{f(x)}{T^n} = \frac{a+n}{n^n}$

Now see only two different x1,x2 exist (x1=X, x2=Y, n is greater than 2)

Assume x1=X, x2=Y such that X > Y, xi = X or Y for i=3,…,n, and $pX + qY = T,\ p + q = n,\ p, q \geq 1$.(it means that $(x1, x2, \ldots, xn) = (X, Y, \ldots X, Y, Y)$ such that x1=x3...=xp+1=X, x2=xp+2=…=xn=Y)

$$\frac{f(X,Y,\ldots X,Y,Y)}{T^n} = \frac{pX^n + qY^n + aX^pY^q}{T^n} = \frac{(1)*X*\frac{p}{n}+(2)*Y*\frac{p}{n}}{T^n} = \frac{\lambda}{nT^{n-1}}$$

So we need to minimum of λ

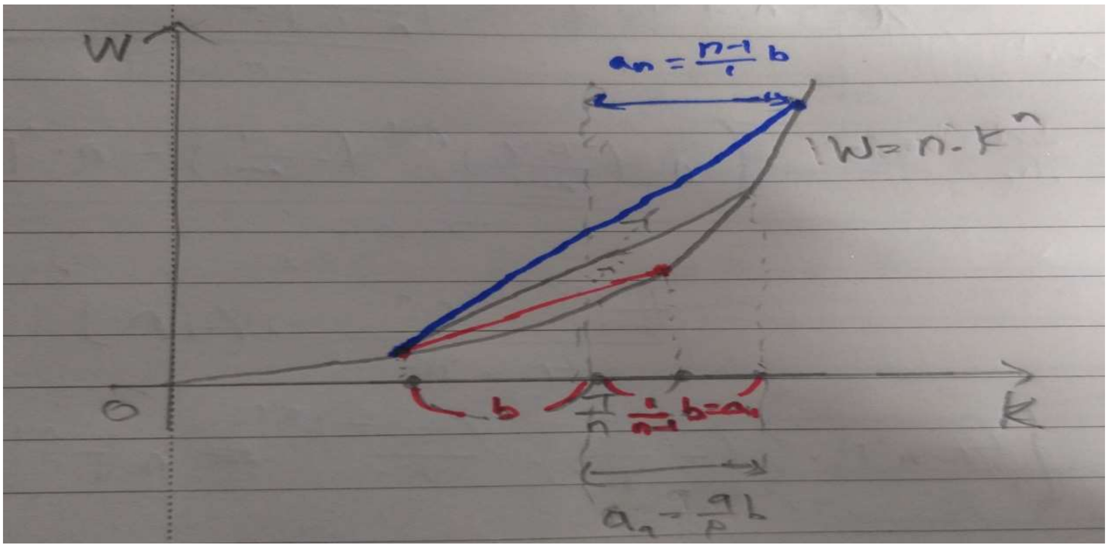
By Using above condition, we can get

X>Y $\Rightarrow$ $\frac{T-qY}{p} > Y$ and $X > \frac{T-pX}{q}$ $\Rightarrow$ $T > X > \frac{T}{n} > Y > 0$

Let X=T/n + a, Y=T/n – b such that pa=qb, a,b>0 $\equiv$ pX+qY = T

(a)*(X-Y) $\Rightarrow$ $\frac{n(X^n - Y^n)}{X-Y} = \lambda$

λ is the gradient of $w = nk^n$.



Although there exist not critical point of p = n-1, q = 1, its λ such that p = n-1, q = 1 and $(n-1)X+Y=T$ is always lower than λ of critical value

So, for a given b, p = n-1, q = 1 to get min λ $(a=\frac{q}{p}b=\frac{1}{n-1}b)$

For (n-1)X+Y=T (p = n-1, q = 1)

$$\lambda=\frac{n(X^n-Y^n)}{X-Y}=n(\left(\frac{T-Y}{n-1}\right)^n-Y^n)/\frac{T-nY}{n-1}$$

$$\frac{d\lambda}{dY}=n(n-1)\left\{\frac{n\left(\frac{T-Y}{n-1}\right)^{n-1}\left(-\frac{1}{n-1}\right)-nY^{n-1}}{T-nY}+\frac{n\left(\left(\frac{T-Y}{n-1}\right)^n-Y^n\right)}{(T-nY)^2}\right\}$$

$$=\frac{n(n-1)}{(T-nY)^2}\{nY\left(\frac{T-Y}{n-1}\right)^{n-1}+Y^{n-1}((n^2-n)Y-nT)\}$$

$$=\frac{(n^2)(n-1)}{(T-nY)^2}Y\{\left(\frac{T-Y}{n-1}\right)^{n-1}+(n-1)Y^{n-1}-TY^{n-2}\}\text{ for }\frac{T}{n}>Y>0$$

As $\frac{(n^2)(n-1)}{(T-nY)^2}Y>0$ for $\frac{T}{n}>Y>0$, Let $h(Y)=\left(\frac{T-Y}{n-1}\right)^{n-1}+(n-1)Y^{n-1}-TY^{n-2}$

$$\frac{dh}{dY}=(n-1)\left(\frac{T-Y}{n-1}\right)^{n-2}\left(-\frac{1}{n-1}\right)+(n-1)^2Y^{n-2}-(n-2)TY^{n-3}$$

$$=-X^{n-2}+(n-1)^2Y^{n-2}-(n-2)\big((n-1)X+Y\big)Y^{n-3}$$

$$=(n^2-3n+3)Y^{n-2}-(n^2-3n+2)XY^{n-3}-X^{n-2}$$

As $Y^{n-2}<XY^{n-3}<X^{n-2}$ for $\frac{T}{n}>Y>0$

$$\frac{dh}{dY}<0\text{ for }\frac{T}{n}>Y>0$$

So $h(Y)$ is decreasing function and $h(0)=\left(\frac{T}{n-1}\right)^{n-1}$ & $h\left(\frac{T}{n}\right)=0.$

$\Rightarrow$ $h(Y)>0$ for $\frac{T}{n}>Y>0$ $\Rightarrow$ $\frac{d\lambda}{dY}=\frac{(n^2)(n-1)}{(T-nY)^2}Yh(Y)>0$ for $\frac{T}{n}>Y>0$

$\Rightarrow$ λ is increasing function and $\lambda(0)=n\left(\frac{T}{n-1}\right)^{n-1}<\lambda(Y)$ for $\frac{T}{n}>Y>0$

$\Rightarrow$ $\frac{\lambda(0)}{nT^{n-1}}=\frac{1}{(n-1)^{n-1}}\leq\min\left(\frac{f(x)}{T^n}\right)=\frac{\lambda}{nT^{n-1}}$ for $\frac{T}{n}>Y>0$

Therefore $\min\left(\frac{f(x)}{T^n}\right)$ of critical points such that two different exist is always greater than $\min\left(\frac{f(x)}{T^n}\right)$ of boundary condition($=\frac{1}{(n-1)^{n-1}}$)

Conclusively, max(b) = $\min(\frac{1}{(n-1)^{n-1}}, \frac{a+n}{n^n})$ for $n \geq 2$

for $n = 1$

$\frac{x1+ax1}{x1} \geq b \Rightarrow \therefore$ max(b)=a+1

for $n \geq 2$

$\therefore$ max(b)=$\frac{1}{(n-1)^{n-1}}$ $for\ a \geq \frac{n^n}{(n-1)^{n-1}} - n$

max(b)= $\frac{a+n}{n^n}$ $for\ a \leq \frac{n^n}{(n-1)^{n-1}} - n$