POW 2021-15 Triangles with integer sides

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Problem : For a natural number n, let a_n be the number of congruence classes of triangles whose all three sides have integer length and its perimeter is n. Obtain a formula for a_n.

Answer) For every nonnegative integer n,

$$\begin{array}{lll} a_{12n} = 3n^2 & a_{12n+1} = 3n^2 + 2n & a_{12n+2} = 3n^2 + n \\ a_{12n+3} = 3n^2 + 3n + 1 & a_{12n+4} = 3n^2 + 2n & a_{12n+5} = 3n^2 + 4n + 1 \\ a_{12n+6} = 3n^2 + 3n + 1 & a_{12n+7} = 3n^2 + 5n + 2 & a_{12n+8} = 3n^2 + 4n + 1 \\ a_{12n+9} = 3n^2 + 6n + 3 & a_{12n+10} = 3n^2 + 5n + 2 & a_{12n+11} = 3n^2 + 7n + 4 \end{array}$$

Solution) Number of congruence classes of triangles is same as number of ordered triple (a, b, c) $(a \geq b \geq c > 0)$ with $a < b + c$. Because two triangles with same side lengths are congruent to each other(SSS congruence), and given positive number $a \geq b \geq c > 0$ with $a < b + c$, it is possible to construct a triangle with side length a, b, c. (In a point of Euclidean geometry view, it is possible to construct a triangle with given such length a, b, c, and in a point of analytic geometry view, a triangle in plane with vertex coordinates $(0, 0)$, $(a, 0)$, $\left(\frac{a^2+b^2-c^2}{2a}, \frac{\sqrt{-(a+b+c)(a+b-c)(a-b+c)(a-b-c)}}{2a}\right)$ has side lengths a, b, c.

So, a_n = number of ordered triple (a, b, c) with $a, b, c \in \mathbb{Z}, a \geq b \geq c > 0$ and $a < b + c$ (or, equivalently, $a < \frac{n}{2}$. Denote $A_n = \{(a, b, c) : a + b + c = n,\ a, b, c \in \mathbb{Z},\ a \geq b \geq c > 0,\ a < \frac{n}{2}\}$. Then $|A_n| = a_n$.

Lemma1) For positive odd integer n, $a_{n+3} = a_n$.

proof) For positive odd integer n, define $f : A_n \to A_{n+3}$ as $f((a, b, c)) = (a + 1, b + 1, c + 1)$. Then, it is well-defined because $a < \frac{n}{2} \Rightarrow (a + 1) < \frac{n+3}{2}$, and it is obviously injective. Also, it is surjective since $(x, y, z) \in A_{n+3}$ implies $x < \frac{1}{2}(n + 3) \Rightarrow (x - 1) < \frac{1}{2}(n + 1) \Rightarrow (x - 1) < \frac{1}{2}n$ (last inequality holds because $n + 1$ is even.) and $z > 1$ because $z = 1$ implies $x > y$ ($\because x + y$ is odd) $\Rightarrow x \geq y + z$. Thus, $(x - 1, y - 1, z - 1) \in A_n$ and $f((x - 1, y - 1, z - 1)) = (x, y, z)$. Therefore f is bijection, and $|A_n| = |A_{n+3}|$.

Lemma2) For positive even integer n, $a_{n+1} = a_n + \lceil \frac{n}{6} \rceil$

proof) For positive even integer n, consider a map $f : A_n \to A_{n+1}$ as $f((a, b, c)) = (a+1, b, c)$. It is well-defined since $a < \frac{n}{2} \Rightarrow a < \frac{n-1}{2} \Rightarrow (a + 1) < \frac{n+1}{2}$.

Injectivity of f is obvious. But f is not surjective. Any element of A_{n+1} of the form (x, x, y) is not in $f(A_n)$, and also any element of $A_{n+1} \backslash f(A_n)$ is of the form (x, x, y). ($\because$ for any element of A_{n+1}, (a, b, c) with $a > b$, $f((a - 1, b, c)) = (a, b, c)$. So, what to show is A_{n+1} has $\lceil \frac{n}{6} \rceil$ elements of the form (x, x, y).

For any element of A_{n+1} of the form (x, x, y) can be expressed as $(\frac{n}{2} - k + 1, \frac{n}{2} - k + 1, 2k - 1)$ for some $k \in \mathbb{Z}$. From condition of A_{n+1}, $1 \leq 2k - 1 \leq \frac{n}{2} - k + 1 \Leftrightarrow 1 \leq k \leq \frac{n}{6} + \frac{2}{3}$. And since n is even, $\lfloor \frac{n}{6} + \frac{2}{3} \rfloor = \lceil \frac{n}{6} \rceil$. Thus, there are $\lceil \frac{n}{6} \rceil$ elements of the form (x, x, y) in A_{n+1}.

Theorem) For every nonnegative integer n,

$$
\begin{aligned}
a_{12n} &= 3n^2 & a_{12n+1} &= 3n^2+2n & a_{12n+2} &= 3n^2+n \\
a_{12n+3} &= 3n^2+3n+1 & a_{12n+4} &= 3n^2+2n & a_{12n+5} &= 3n^2+4n+1 \\
a_{12n+6} &= 3n^2+3n+1 & a_{12n+7} &= 3n^2+5n+2 & a_{12n+8} &= 3n^2+4n+1 \\
a_{12n+9} &= 3n^2+6n+3 & a_{12n+10} &= 3n^2+5n+2 & a_{12n+11} &= 3n^2+7n+4
\end{aligned}
$$

proof) Using induction. For the base case, $a_0 = a_1 = a_2 = 0$ and it satisfies the statement. Now consider a_n for $n > 2$.

From Lemma1, $a_{12n} = a_{12(n-1)+9} = 3(n-1)^2 + 6(n-1) + 3 = 3n^2$.

From Lemma2, $a_{12n+1} = a_{12n+1} + \lceil \frac{12n}{6} \rceil = 3n^2 + 2n$.

From Lemma1, $a_{12n+2} = a_{12(n-1)+11} = 3(n-1)^2 + 7(n-1) + 4 = 3n^2 + n$.

From Lemma2, $a_{12n+3} = a_{12n+2} + \lceil \frac{12n+2}{6} \rceil = (3n^2 + n) + (2n+1) = 3n^2 + 3n + 1$.

From Lemma1, $a_{12n+4} = a_{12n+1} = 3n^2 + 2n$.

From Lemma2, $a_{12n+5} = a_{12n+4} + \lceil \frac{12n+4}{6} \rceil = (3n^2 + 2n) + (2n+1) = 3n^2 + 4n + 1$.

From Lemma1, $a_{12n+6} = a_{12n+3} = 3n^2 + 3n + 1$.

From Lemma2, $a_{12n+7} = a_{12n+6} + \lceil \frac{12n+6}{6} \rceil = (3n^2 + 3n + 1) + (2n+1) = 3n^2 + 5n + 2$.

From Lemma1, $a_{12n+8} = a_{12n+5} = 3n^2 + 4n + 1$.

From Lemma2, $a_{12n+9} = a_{12n+8} + \lceil \frac{12n+8}{6} \rceil = (3n^2 + 4n + 1) + (2n+2) = 3n^2 + 6n + 3$.

From Lemma1, $a_{12n+10} = a_{12n+7} = 3n^2 + 5n + 2$.

From Lemma2, $a_{12n+11} = a_{12n+10} + \lceil \frac{12n+10}{6} \rceil = (3n^2 + 5n + 2) + (2n+2) = 3n^2 + 7n + 4$.