Problem. Prove or disprove the following:

There exist an infinite sequence of \( f_n : [0, 1] \rightarrow \mathbb{R} \), \( n = 1, 2, 3, \ldots \), such that

1. \( f_n(0) = f_n(1) = 0 \) for any \( n \)
2. \( f_n \left( \frac{a+b}{2} \right) \leq f_n(a) + f_n(b) \) for any \( a, b \in [0, 1] \)
3. \( f_n - cf_m \) is not identically zero for any \( c \in \mathbb{R} \) and \( n \neq m \).

Solution. The answer is, **exist**. Let's construct such \( f_n \)’s.

Let \( V \) be a set of such functions; i.e.,

\[
V = \left\{ f : [0, 1] \rightarrow \mathbb{R} \mid f(0) = f(1) = 0, \forall a, b \in [0, 1], f \left( \frac{a+b}{2} \right) \leq f(a) + f(b) \right\}.
\]

One can easily show that:

i) if \( f, g \in V \), then \( f + g \in V \), and

ii) if \( f \in V \), \( \lambda \in \mathbb{R}_{>0} \) then \( \lambda f \in V \).

Suppose we have two independent functions \( g_1, g_2 \in V \), define \( f_n = n \cdot g_1 + g_2 \in V \).

(Here, \( g_1, g_2 \) are independent means that \( g_1 \neq 0 \), \( g_2 \neq 0 \),
and there is no \( c \in \mathbb{R} \) s.t. \( g_1 = c \cdot g_2 \).)

Then, \( f_n \) and \( f_m \) are independent for \( n \neq m \).

Hence, it is enough to find two independent functions \( g_1, g_2 \in V \).

First, let’s consider necessary conditions for \( g \in V \).
1. For \( x \in [0,1] \), take \( a = b = x \) and (2) implies
\[ g \left( \frac{a+b}{2} \right) \leq g(a) + g(b) \implies g(x) \geq 0. \]

2. \[ g \left( \frac{1}{2} \right) \leq g(0) + g(1) = 0, \]
\[ g \left( \frac{1}{4} \right) \leq g(0) + g(\frac{1}{2}) \leq 0, \quad g \left( \frac{3}{4} \right) \leq g(\frac{1}{2}) + g(1) \leq 0, \]
and so on \( g \left( \frac{1}{2^k} \right) \leq 0, \quad g \left( \frac{1}{2^{k+1}} \right) \leq 0 \).

One can show that
\[ g \left( \frac{1}{2^k} \right) \leq 0 \]
for nonnegative integer \( k \) and positive odd integer \( l \), \( 1 \leq l < 2^k \),
by induction on \( k \). Since \( g \geq 0 \), \( g \left( \frac{l}{2^k} \right) = 0 \).

Let \( A = \left\{ \frac{l}{2^k} \mid k, l \in \mathbb{Z}_{\geq 0}, \quad 0 \leq l \leq 2^k \right\} \subset [0,1] \).
So, \( g(x) = 0 \) for \( x \in A \), and we have to assign values of \( g(x) \) for \( x \in [0,1] \setminus A \).

Define \( g_1(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in [0,1] \setminus A \end{cases} \).

We have to show \( g_1 \) satisfies the condition (2).

Equivalently, write (2) as
\[ g_1(x) \leq g_1(x-h) + g_1(x+h) \]
for \( x \in [0,1] \), \( 0 \leq h \leq \min \{x, 1-x\} \).
— If \( x \in A \), the inequality holds obviously.
— Suppose \( x \in [0,1] \setminus A \). Note that \( \frac{x+h}{2} \in A \) if \( a, b \in A \);
  hence, at least one of \( x-h, x+h \) is in \( [0,1] \setminus A \),
  so the inequality holds.

So far, we've found \( g_1 \neq 0 \).
For another one, recall the binary expansion: $\pi \in [0,1)$ has a unique binary expansion

$$
\pi = 0.\ a_1 a_2 a_3 \cdots \ (2) = \sum_{j=1}^{\infty} \frac{a_j}{2^j}, \quad a_j \in \{0,1\}.
$$

For the uniqueness, we do not allow the situation that the sequence $\{a_j\}$ is "eventually 1".

Define $\delta_2(x) = \liminf_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} a_j$ for $x \in [0,1)$, and $\delta_2(0) = 0$.

\[ x = 0.\ a_1 a_2 \cdots \ (2), \quad y = 0.\ b_1 b_2 \cdots \ (2), \quad \text{and} \]
\[ \frac{1}{2}(x+y) = 0.\ a_1 a_2 \cdots \ (2) + 0.\ b_1 b_2 \cdots \ (2) = 0.\ c_1 c_2 \cdots \ (2). \]

For the condition (2), ETS
\[ \sum_{j=1}^{N} c_j \leq \sum_{j=1}^{N} (a_j + b_j), \]
and this is true since carry decreases the total summation of digits.

(For example (allow the following weird notation for a while):)
\[ x+y \ | \ 1011_{(2)} + 11_{(2)} = 1022_{(2)} \rightarrow 1030_{(2)} \rightarrow 1110_{(2)} \]
\[ \sum c_j \ | \ 3 + 2 = 5 \rightarrow 4 \rightarrow 3 \]

Of course, there are cases that $\{c_j\}$ is eventually 1, but again to make these 1's to 0's decreases the total summation of digits.
\[ 0.\ \cdots \ 0111 \cdots \ (2) \rightarrow 0.\ \cdots \ 1000 \cdots \ (2). \]

Hence
\[ \frac{1}{N} \sum_{j=1}^{N} c_j \leq \frac{1}{N} \sum_{j=1}^{N} a_j + \frac{1}{N} \sum_{j=1}^{N} b_j, \]
and
\[ g\left(\frac{x+y}{2}\right) = \liminf_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} c_j \leq \liminf_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} a_j + \liminf_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} a_j \]
\[ = g_2(x) + g_2(y). \]
And 
\[
\frac{1}{3} = \sum_{j=1}^{\infty} \left( \frac{1}{4} \right)^j = 0.1010101\ldots_{(2)}, \quad g_2(\frac{1}{3}) = \frac{1}{3},
\]
\[
\frac{1}{7} = \sum_{j=1}^{\infty} \left( \frac{1}{8} \right)^j = 0.001001001\ldots_{(2)}, \quad g_2(\frac{1}{7}) = \frac{1}{3} \neq g_2(\frac{1}{3}),
\]

thus \( g_1, g_2 \neq 0 \) are independent. Done! \( \square \)