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Problem Prove or disprove the following:

There exist an infinite sequence of $f_n : [0,1] \to \mathbb{R}$, $n = 1, 2, 3, \cdots$, such that

(1) $f_n(0) = f_n(1) = 0$ for any n

(2) $f_n\left(\frac{a+b}{2}\right) \le f_n(a) + f_n(b)$ for any $a, b \in [0,1]$,

(3) $f_n - c f_m$ is not identically zero for any $c \in \mathbb{R}$ and $n \le m$.

Solution The answer is, EXIST. Let's construct such f_n's.

Let V be a set of such functions; i.e.

$$V = \left\{ f : [0,1] \to \mathbb{R} \;\middle|\; f(0) = f(1) = 0,\ \forall a, b \in [0,1]\ f\left(\tfrac{a+b}{2}\right) \le f(a) + f(b) \right\}.$$

One can easily show that:

i) if $f, g \in V$, then $f + g \in V$, and

ii) if $f \in V$, $\lambda \in \mathbb{R}_{\ge 0}$ then $\lambda f \in V$.

Suppose we have two independent functions $g_1, g_2 \in V$, define $f_n = n \cdot g_1 + g_2 \in V$.

(Here, g_1, g_2 are <u>independent</u> means that $g_1 \not\equiv 0$, $g_2 \not\equiv 0$, and there is no $c \in \mathbb{R}$ s.t. $g_1 \equiv c \cdot g_2$.)

Then, f_n and f_m are independent for $n \ne m$.

Hence, it is enough to find two independent functions $g_1, g_2 \in V$.

First, let's consider necessary conditions for $g \in V$.

⊙ For $x \in [0,1]$, take $a=b=x$ and (2) implies

$$g\left(\frac{x+x}{2}\right) \le g(x) + g(x) \Rightarrow g(x) \ge 0.$$

⊙ $g\left(\frac{1}{2}\right) \le g(0)+g(1) = 0$,

$g\left(\frac{1}{4}\right) \le g(0)+g\left(\frac{1}{2}\right) \le 0$, $g\left(\frac{3}{4}\right) \le g\left(\frac{1}{2}\right)+g(1) \le 0$,

and so on $g\left(\frac{1}{8}\right) \le 0$, $g\left(\frac{3}{8}\right) \le 0$, $\cdots$.

One can show that

$$g\left(\frac{l}{2^k}\right) \le 0$$

for nonnegative integer k and positive odd integer l, $1 \le l < 2^k$, by induction on k. Since $g \ge 0$, $g(l/2^k) = 0$.

Let $A = \left\{ \frac{l}{2^k} \mid k, l \in \mathbb{Z}_{\ge 0},\ 0 \le l \le 2^k \right\} \subset [0,1]$.

So, $g(x) = 0$ for $x \in A$, and we have to assign values of $g(x)$ for $x \in [0,1] \setminus A$.

Define
$$g_1(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in [0,1] \setminus A \end{cases}.$$

We have to show g_1 satisfies the condition ②.

Equivalently, write ② as

$$g_1(x) \le g_1(x-h) + g_1(x+h)$$

for $x \in [0,1]$, $0 \le h \le \min\{x, 1-x\}$.

— If $x \in A$, the inequality holds obviously.

— Suppose $x \in [0,1] \setminus A$. Note that $\frac{a+b}{2} \in A$ if $a, b \in A$; hence, at least one of $x-h$, $x+h$ is in $[0,1] \setminus A$, so the inequality holds.

So far, we've found $g_1 \not\equiv 0$.

For another one, recall the binary expansion: $x \in [0,1)$ has a unique binary expansion

$$x = 0.a_1 a_2 a_3 \cdots_{(2)} = \sum_{j=1}^{\infty} \frac{a_j}{2^j}, \quad a_j \in \{0,1\}.$$

For the uniqueness, we do not allow the situation that the sequence $\{a_j\}$ is "eventually 1".

Define $g_2(x) = \liminf_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} a_j$ for $x \in [0,1)$, and $g_2(1) = 0$.

Let $x = 0.a_1 a_2 \cdots_{(2)}$, $y = 0.b_1 b_2 \cdots_{(2)}$, and

$$\tfrac{1}{2}(x+y) = 0.0\,a_1 a_2 \cdots_{(2)} + 0.0\,b_1 b_2 \cdots_{(2)} = 0.c_1 c_2 \cdots_{(2)}.$$

For the condition (2), ETS

$$\sum_{j=1}^{N} c_j \le \sum_{j=1}^{N} (a_j + b_j),$$

and this is true, since carry decreases the total summation of digits.

For example (allow the following weird notation for a while):

$$\begin{array}{c|l} x+y & 1011_{(2)} + 11_{(2)} = 1022_{(2)} \xrightarrow{\text{carry}} 1030_{(2)} \xrightarrow{\text{carry}} 1110_{(2)} \\ \sum c_j & 3 + 2 = 5 \longrightarrow 4 \longrightarrow 3 \end{array}$$

Of course, there are cases that $\{c_j\}$ is eventually 1, but again to make these 1's to 0's decreases the total summation of digits.

$$0.\cdots 0111\cdots_{(2)} \to 0.\cdots 1000\cdots_{(2)}$$

Hence $\frac{1}{N}\sum_{j=1}^{N} c_j \le \frac{1}{N}\sum_{j=1}^{N} a_j + \frac{1}{N}\sum_{j=1}^{N} b_j$, and

$$g\left(\frac{x+y}{2}\right) = \liminf_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} c_j \le \liminf_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} a_j + \liminf_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} a_j$$
$$= g_2(x) + g_2(y).$$

And

$$\frac{1}{3} = \sum_{j=1}^{\infty} \left(\frac{1}{4}\right)^j = 0.010101\cdots_{(2)}, \quad g_2\left(\frac{1}{3}\right) = \frac{1}{2},$$

$$\frac{1}{7} = \sum_{j=1}^{\infty} \left(\frac{1}{8}\right)^j = 0.001001\cdots_{(2)}, \quad g_2\left(\frac{1}{7}\right) = \frac{1}{3} \neq g_2\left(\frac{1}{3}\right),$$

thus $g_1, g_2 \not\equiv 0$ are independent. Done!

$\blacksquare$