pow2021-12

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June 2021

Problem. *In a graduation ceremony, n graduating students form a circle and their diplomas are distributed uniformly at random. Students who have their own diploma leave, and each of the remaining students passes the diploma she has to the student on her right, and this is one round. Again, each student with her own diploma leave and each of the remaining students passes the diploma to the student on her right and repeat this until everyone leaves. What is the probability that this process takes exactly k rounds until everyone leaves?*

Solution. Let n be fixed. For the simplicity, we change the rule by following: after student got her own diploma, instead of leaving she stays and at each passing process she keeps her own diploma and passes the diploma she got from her left to her right. The process ends if every student got her own diploma. Obviously even under the changed rule each student gets the same diploma at each round as in the original one. Also, distribution of diploma can be treated as permutation σ on $[n]$. When $(i+1)$-th student is on right of i-th student for $i = 1, 2, \cdots, n-1$ and n-th student is on the left of first student, let $\sigma \in S_n$ be defined as $\sigma(i) = j$ if j-th student has the diploma of i-th student. Then our goal is to find out the number of permutations which become identity map after the k iterations of transformation T which corresponds to the passing process. To do so we start with giving two definitions: one is assigning value(which is in some sense similar to displacement of permutation) to each $\sigma \in S_n$, and the other is transformation on S_n which corresponds to the passing process.

Definition (Modified displacement)**.** *For $\sigma \in S_n$, the modified displacement $d(\sigma)$ is defined as:*

$$d(\sigma) = \sum_{\sigma(i) \le i} (i - \sigma(i)) + \sum_{\sigma(i) > i} (n - \sigma(i) + i)$$

One can easily deduce that in fact $d(\sigma) = nk$ where $k = \#\{i \in [n] : \sigma(i) > i\}$, and identity map is the only permutation whose modified displacement is zero. The term in summation($(i - \sigma(i))$ or $(n - \sigma(i) + i)$) can be interpreted as the number of students other than i-th student who once had a diploma of i in the ceremony, including the immediate passing after receiving it. This can be seen as distance between the diploma and its owner, and this is why we named d as modified displacement. Now we define transformation $T : S_n \to S_n$ which represents the passing process.

Definition (Passing process)**.** *For $\sigma \in S_n$, let F_σ be the set of fixed points of σ. Then $T\sigma \in S_n$ is defined as:*

$$(T\sigma)(i) = \begin{cases} i, & \text{if } \sigma(i) \in F_\sigma \\ \min((F_\sigma)^{\mathsf{c}}), & \text{if } \sigma(i) = \max((F_\sigma)^{\mathsf{c}}) \\ \min(\{\sigma(i)+1, \sigma(i)+2, \cdots, n\} \cap (F_\sigma)^{\mathsf{c}}), & \text{if } \sigma(i) \in (F_\sigma)^{\mathsf{c}} \text{ and } \sigma(i) \neq \max((F_\sigma)^{\mathsf{c}}) \end{cases}$$

This indeed fits to the passing process described above. At the beginning of each round $\sigma(i)$-th student has i-th student's diploma, and she passes it to the $(\sigma(i) + 1)$-th student, who is on the right of her. If $(\sigma(i)+1)$-th student already got her own diploma at the end of last round, she will pass i-th student's diploma again. Otherwise, she will keep it. This procedure will end when the receiving student does not have her own diploma. At the end of round $T\sigma$ will be the distribution of diploma. Then we have following lemma:

Lemma 1. *For any $\sigma \in S_n$ except the identity, $d(T\sigma) = d(\sigma) - n$.*

Proof. For each $i \in [n]$, if i is fixed point, $(T\sigma)(i) = \sigma(i) = i$. If $\sigma(i) \in (F_\sigma)^{\mathsf{c}}$ and $\sigma(i) \neq \max((F_\sigma)^{\mathsf{c}})$, $(T\sigma)(i) > i$ if $\sigma(i) > i$ and $(T\sigma)(i) \leq i$ if $\sigma(i) < i$ by definition. If $\sigma(i) = \max((F_\sigma)^{\mathsf{c}})$, which is greater than i by its definition, since $(T\sigma)(i) = \min((F_\sigma)^{\mathsf{c}})$ and $i \in (F_\sigma)^{\mathsf{c}}$ we have $(T\sigma)(i) \leq i$. Thus

$$\#\{i \in [n] : (T\sigma)(i) > i\} = \#\{i \in [n] : \sigma(i) > i\} - 1$$

and this completes the proof. Note that there exists another proof which is more intuitive: At each passing process, if the diploma have already arrived to its owner, it doesn't move any more. However the sum of the others getting closer to their owner is increased by $|F_\sigma|$ thus in total the sum of distances between the diploma and their owner decreases by n. □

Using Lemma 1 we can deduce that the ceremony ends after $k+1$ rounds if and only if $\#\{i \in [n] : \sigma(i) > i\} = k$ where σ denotes the initial distribution of diploma. Thus the probability $p(n,k)$ of ceremony ends after k rounds is

$$p(n,k) = \frac{\#\{\sigma \in S_n : \#\{i \in [n] : \sigma(i) > i\} = k-1\}}{|S_n|} = \frac{A(n,k-1)}{n!}$$

where $A(n,k-1)$ is an Eulerian number. □