KAIST Problem of the Week 2021-11

Hang Park (2013____; graduate of CS and MAS)
hangpark@kaist.ac.kr

Let $d(n)$ be a sum of digits of a natural number n with base 10, i.e., if $n = \sum_{k=0}^{l} 10^k a_k$ with non-negative integers l and $0 \leq a_k \leq 9$ such that $a_l \neq 0$, then $d(n)$ is defined by $d(n) = \sum_{k=0}^{l} a_k$. Note that any $n \in \mathbb{N}$ can be uniquely expressed by l and a_ks.

Lemma 1. *$d(n) \equiv n \pmod 9$ for all $n \in \mathbb{N}$.*

Proof. Let $n = \sum_{k=0}^{l} 10^k a_k$ as shown above. Then,

$$n = \sum_{k=0}^{l} 10^k a_k = \sum_{k=0}^{l} a_k + \sum_{k=0}^{l} (10^k - 1) a_k \equiv \sum_{k=0}^{l} a_k = d(n) \pmod 9$$

since $9 \mid 10^k - 1$ for each k. □

Lemma 2. *$d(n) < 9 + 9\log_{10} n$ for all $n \in \mathbb{N}$.*

Proof. Let $n = \sum_{k=0}^{l} 10^k a_k$ as shown above. Then, $10^l \leq n$ gives $l \leq \log_{10} n$ with equality on $10^l = n$. Since $a_k \leq 9$, for $n \neq 10^l$, we have

$$d(n) = \sum_{k=0}^{l} a_k \leq \sum_{k=0}^{l} 9 = 9(1+l) < 9 + 9\log_{10} n.$$

For $n = 10^l$, $d(n) = 1 < 9 \leq 9 + 9\log_{10} n$. □

Problem. *Find all natural numbers n such that $d(n^3) = n$.*

Solution. Suppose that $d(n^3) = n$. By Lemma 1 we have $n = d(n^3) \equiv n^3 \pmod 9$, thus, $n \equiv 0 \pmod 9$ or $n \equiv \pm 1 \pmod 9$ since $n^3 - n = (n-1)n(n+1) \equiv 0 \pmod 9$.

Note that $d(100^3) < 9 + 9\log_{10} 100^3 = 63 < 100$ by Lemma 2. Since the derivative of $x - (9 + 9\log_{10} x^3)$ is $1 - 27/x$ hence the subtraction increases on $x \geq 100 > 27$, we can know that $d(n^3) < n$ for all $n \geq 100$, thus n should be less than 100.

For $n < 100$, $d(n^3) < 9 + 9\log_{10} n^3 < 9 + 9\log_{10} 100^3 = 63$ holds, thus $n < 63$. So possible candidates are 1, 8, 9, 10, 17, 18, 19, 26, 27, 28, 35, 36, 37, 44, 45, 46, 53, 54, and 55. Among theses, simple calculation gives only 1, 8, 17, 18, 26, and 27 satisfy $d(n^3) = n$. □