POW 2021-10 Integral inequality

Kee Taek Kim

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Problem. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function satisfying

$$\int_x^1 f(t)\,dt \geq \int_x^1 t\,dt$$

for all $x \in [0,1]$. Prove that

$$\int_0^1 [f(t)]^2 \, dt \geq \int_0^1 tf(t)\,dt.$$

Solution. Let

$$F(x) = \int_x^1 [f(t) - t] \, dt \geq 0,$$

then $F'(t) = t - f(t)$, and $F(1) = 0$. We want to show that

$$\int_0^1 \left[f(t)^2 - tf(t)\right] \, dt = \int_0^1 [t - F'(t)]\,[-F'(t)] \, dt$$

is equal or greater than 0. By the integration by parts,

$$\int_0^1 [-tF'(t)] \, dt = [-tF(t)]_0^1 + \int_0^1 F(t)\,dt = \int_0^1 F(t)\,dt \geq 0.$$

Hence

$$\int_0^1 [t - F'(t)]\,[-F'(t)] \, dt = \int_0^1 [-tF'(t)] \, dt + \int_0^1 [F'(t)]^2 \, dt \geq 0$$

and the equality holds when $[F'(t)]^2 = 0$, i.e. $f(t) = t$.