Problem. Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function satisfying
\[
\int_x^1 f(t) \, dt \geq \int_x^1 t \, dt
\]
for all \( x \in [0, 1] \). Prove that
\[
\int_0^1 [f(t)]^2 \, dt \geq \int_0^1 tf(t) \, dt.
\]

Solution. Let
\[
F(x) = \int_x^1 [f(t) - t] \, dt \geq 0,
\]
then \( F'(t) = t - f(t) \), and \( F(1) = 0 \). We want to show that
\[
\int_0^1 \left[ f(t)^2 - tf(t) \right] \, dt = \int_0^1 t [-F'(t)] \, dt
\]
is equal or greater than 0. By the integration by parts,
\[
\int_0^1 [-tF'(t)] \, dt = [-tF(t)]_0^1 + \int_0^1 F(t) \, dt = \int_0^1 F(t) \, dt \geq 0.
\]
Hence
\[
\int_0^1 \left[ t - F'(t) \right] [-F'(t)] \, dt = \int_0^1 [-tF'(t)] \, dt + \int_0^1 [F'(t)]^2 \, dt \geq 0
\]
and the equality holds when \([F'(t)]^2 = 0\), i.e. \( f(t) = t \).