POW 2021-09

Anon.

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The answer is $n = k^2 + k - 1$.

First, we can color numbers in $[n-1] = [k^2 + k - 2]$ so that there are no $x_0, \cdots, x_k$ of the same color satisfying $x_1 + \cdots + x_k = x_0$: Color 1 to $k - 1$ as red, k to $k^2 - 1$ as blue, k^2 to $k^2 + k - 2$ as red.
It is obvious that the number in the first block cannot be expressed as the sum of k numbers and the number in the second block cannot be expressed as the sum of k blue-colored numbers. For the final block, as $k^2 + k - 1 = k^2 + (k-1) \times 1$ is the smallest sum of the k red-colored numbers that exceed the first block range, so the same holds.

Now we see why it is impossible to color numbers in $[n] = [k^2 + k - 1]$ without satisfying the condition. Let us try to avoid the condition.

- Let us say 1 is colored as red.
- k must be colored as blue because $k = k \times 1$.
- k^2 must be colored as red because $k^2 = k \times k$.
- $k^2 + k - 1$ must be colored as blue because $k^2 + k - 1 = k^2 + (k-1) \times 1$.
- $k + 1$ must be colored as blue because $k^2 = (k-1) \times (k+1) + 1$.
- Now $k^2 + k - 1 = (k-1) \times (k+1) + k$, here, the condition is satisfied.

Therefore, it is impossible to avoid the condition.