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Joonhyung Shin

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We first show that each (connected) component of $S^2 \setminus C$ does not contain antipodal points. Suppose not: assume that $\mathbf{x}, -\mathbf{x} \in X$ for some component $X$ of $S^2 \setminus C$. Since $X$ is open and connected, it is path-connected, so there is a path from $\mathbf{x}$ to $-\mathbf{x}$. Concatenating the path with its image under the antipodal map, we obtain a self-antipodal loop $\gamma : S^1 \to S^2$ based on $\mathbf{x}$. Take any $\mathbf{a} \in C$ and let $Y$ be the (connected) component of $S^2 \setminus \gamma$ containing $\mathbf{a}$. Since $C \subseteq S^2 \setminus \gamma$ and $C$ is connected, $Y$ should contain $-\mathbf{a}$ as well.

Since $Y$ is open, it is path-connected. Thus, let $P$ be a path from $\mathbf{a}$ to $-\mathbf{a}$. Embed $S^2$ into $\mathbb{R}^3$ and without loss of generality assume further that $\mathbf{a} = (0, 0, 1)$. Let $\varphi : S^2 \setminus \{\mathbf{a}, -\mathbf{a}\} \to S^1$ be a "projection" map given by $(x, y, z) \mapsto (x/\sqrt{x^2 + y^2}, y/\sqrt{x^2 + y^2})$.

Since $P$ is Hausdorff and is a continuous image of the unit interval, $P$ is arc-connected ([1], Theorem 31.2). Hence, we may assume that $P$ is an embedding of the unit interval. Then $S^2 \setminus P$ has trivial reduced homologies ([2], Proposition 2B.1), so the map $\varphi \circ \gamma : S^1 \to S^2 \setminus P \to S^1$ induces the trivial map on the first homology. But since $\varphi \circ \gamma$ is an odd map, its degree is odd so it cannot induce the trivial map ([2], Proposition 2B.6). This is a contradiction.

Therefore, each component of $S^2 \setminus C$ does not contain antipodal points. These components appear in pairs: if $X$ is a component, then so is $-X = \{-\mathbf{x} : \mathbf{x} \in X\}$. Thus, our goal is to construct a smooth function $f_X$ supported by $X$, set $f_{-X} = -f_X$, and sum $f_X + f_{-X}$ over all (antipodal) component pairs.

First, we construct a smooth function $f_X$ such that $f_X > 0$ on $X$ and $f_X = 0$ on $S^2 \setminus X$. Since $X$ is open and not the whole space, we may work on $\mathbb{R}^2$ in lieu of $S^2$, where $X$ is assumed to be open and have compact closure. This can be done by a standard technique using mollifiers. Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$\psi(\mathbf{x}) = \begin{cases} e^{-\frac{1}{1-x_1^2} - \frac{1}{1-x_2^2}} & \mathbf{x} \in (-1, 1) \times (-1, 1) \\ 0 & \text{otherwise} \end{cases}. \tag{1}$$

Also define $\psi_n(\mathbf{x}) = \psi(n\mathbf{x})$ and

$$V_n = \left\{ \mathbf{x} \in \mathbb{R}^2 : d(\mathbf{x}, \mathbb{R}^2 \setminus X) > \frac{1}{n} \right\} . \tag{2}$$

Let $f_n : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f_n(\mathbf{x}) = \frac{1}{2^n} (\chi_{V_n} * \psi_n)(\mathbf{x})$$

where $\chi_{V_n}$ is the characteristic function of $V_n$, and finally

$$f_X(\mathbf{x}) = \sum_{n=1}^{\infty} f_n(\mathbf{x}) .$$

The series converges uniformly in all derivaties (by, for instance, the Weierstrass $M$-test) so the resulting function is smooth.

Now we sum up the functions $f_X + f_{-X}$ as we planned, but care should be taken since there may be (countably) infinite number of components of $S^2 \setminus X$. One way of handling this is to adjust the mollifier (1) and the open sets (2) so that the metric is given by the standard metric on $S^2$ (instead of that on $\mathbb{R}^2$). Then our construction guarantees the uniform convergence of the series $f = \sum (f_X + f_{-X})$ in all derivatives, thereby ensuring the smoothness of the resulting function $f$.

# References


[1] Stephen Williard. *General Topology.* Dover Publications, 2012.

[2] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2001.