2021-07 Odd determinant

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Problem.

Let A_N be an $N \times N$ matrix whose entries are i.i.d. Bernoulli random variables with probability $1/2$, i.e.,

$$\mathbb{P}((A_N)_{ij} = 0) = \mathbb{P}((A_N)_{ij} = 1) = \frac{1}{2}.$$

Let p_N be the probability that $\det A_N$ is odd. Find $\lim_{N\to\infty} p_N$.

Solution.

A_N having odd determinant is equivalent with A_N having non-zero determinant, if we consider A_N over $\mathbb{Z}_2$. This is also equivalent to A_N is invertible, and set of row vector is linearly independent.

Let's calculate the number of invertible $N \times N$ matrices over $\mathbb{Z}_2$. We will choose $(i+1)$'th row vector $\vec{v}_{i+1}$ as one not included in $S = span\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_i}\}$. Each element of S can be uniquely represented as $\sum_{j=1}^{i} c_j \vec{v_j}$ where $c_j \in \mathbb{Z}_2$, so $|S| = 2^i$ and $|S^c| = |\mathbb{Z}_2^n \setminus S| = 2^N - 2^i$. Using rule of product, the number of invertible matrices is $\prod_{i=0}^{N-1} (2^N - 2^i)$. p_N is $\prod_{i=1}^{N} (1 - 2^{-i})$ and $\lim_{N\to\infty} p_N = \prod_{i=1}^{\infty} (1 - 2^{-i}) = 0.288788095\cdots$.