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Let $l(k)$ be the least positive integer such that for any integer $n \geq l(k)$ every element of $\mathcal{A}_n$ has nondecreasing subsequence of length greater than or equal to k. It is clear that $l(1) = 1$, $l(2) = 2$, $l(3) = 4$. From these observations one may hope that $l(k+1) = 2^k$ for all $k \geq 0$. We show this thought is indeed true via induction on k. Assume $l(m) = 2^{m-1}$ for all $m \leq k$. Then we have useful lemma:

Lemma 1. *Let $\{a_i\}_{i\in[n]}$ be element of $\mathcal{A}_n$ whose nondecreasing subsequence is length of at most k. Then 1 appears in $\{a_i\}_{i\in[n]}$ at most k times, and any number greater than 2^m appears in $\{a_i\}_{i\in[n]}$ at most $k-m-1$ times, for $m = 0, 1, 2, \cdots, k-1$.*

Proof. The first argument is trivial. For the second argument, assume $x > 2^m$ appears in $\{a_i\}_{i\in[n]}$ more than $k-m-1$ times. Since $\{a_i\}_{i\in[n]} \in \mathcal{A}_n$ existence of x implies $n > 2^m$ and $\{a_i\}_{i\leq 2^m}$ contains nondecreasing subsequence of length $m+1$, while the last term is less than or equal to 2^m. Moreover $\{a_i\}_{i\leq 2^m}$ contains no x. Thus by joining x to the end of such subsequence $k-m$ times we get nondecreasing subsequence of length $k+1$, which is contradiction. $\square$

Thus if maximal length of nondecreasing subsequence of $\{a_i\}_{i\in[n]}$ is k, 1 appears at most k times, 2 appears at most $k-1$ times, $3, 4$ appears at most $k-2$ times, and so on. Therefore we get

$$n \leq k + (k-1)(2^1 - 2^0) + \cdots + (2^{k-1} - 2^{k-2}) = 2^k - 1,$$

and if $n \geq 2^k$, $\{a_i\}_{i\in[n]}$ has nondecreasing subsequence of length $k+1$, which implies $l(k+1) \leq 2^k$. We claim this bound is optimal, by suggesting following sequence: Consider ordered sets

$$I_k = \{2^k, 2^k - 1, 2^k - 2, \cdots, 3, 2, 1\}, k \in \mathbb{Z}_{\geq 0}.$$

Then ordered sum S_{k-1} of $I_0, I_1, I_2, \cdots, I_{k-1}$ is:

$$S_{k-1} = \{1, 2, 1, 4, 3, 2, 1, 8, 7, 6, 5, 4, 3, 2, 1, 16, 15, \cdots, 2^{k-1}, 2^{k-1} - 1, \cdots, 2, 1\}$$

The length of S_{k-1} is $2^k - 1$. Since each I_i is decreasing, any nondecreasing subsequence of S_{k-1} cannot contain more than element of I_i for $i = 0, 1, 2, \cdots, k-1$. Thus its length is at most k and $l(k+1) > 2^k - 1$, which gives $l(k+1) = 2^k$.