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Finite generation of a group

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Let $k = \mathbb{F}_2$ be the field with two elements, namely 0 and 1. Claim that the ring of polynomials $k[x]$ on one variable x as an additive group gives a counterexample for the statement. Clearly, as we have $x^n \in k[x]$ for $n \in \mathbb{Z}_{\geq 0}$, $k[x]$ is an infinite group. Observe that for each element $\sum_{i=0}^{n} c_i x^i \in k[x]$ we have

$$\left(\sum_{i=0}^{n} c_i x^i\right) + \left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=0}^{n} 2 \cdot c_i x^i = \sum_{i=0}^{n} 0 \cdot x^i = 0$$

hence every element of $k[x]$ has degree at most 2, in particular less than 3. $k[x]$ satisfies the assumptions of the statement.

Let S be a finite subset of $k[x]$. Exploiting the fininteness, there exists an integer $n \geq 0$ such that every element of S has degree at most n. Every element of $\langle S \rangle$ has degree at most n. Therefore, $x^{n+1} \notin \langle S \rangle$ and S does not generate $k[x]$. As we cannot choose any finite subset of $k[x]$ that generates $k[x]$, it is not finitely generated as an additive group.