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Ko Sunghun

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Problem. *For any positive semideifinite A and B, prove that* $[\lambda(A^m B^m)]^{\frac{1}{m}} \le [\lambda(A^{m+1}B^{m+1})]^{\frac{1}{m+1}}$.

We prove useful lemma firstly and proceed by induction on m. Since A and B are both hermitian, there exist unitary matrices U, V and diagonal matrices D_1, D_2 such that $A = UD_1U^*, B = VD_2V^*$. Moreover, diagonal entries of D_1 and D_2 are nonnegative, thus $\sqrt{D_1}$ and $\sqrt{D_2}$ are well defined. Note that from these definitions we may define $\sqrt{A}, \sqrt{B}$ as $\sqrt{A} = U\sqrt{D_1}U^*, \sqrt{B} = V\sqrt{D_2}V^*$, too. We have following lemma:

Lemma 1. *for any $\mu \in \mathbb{C}\setminus\{0\}$ and $p, q, r \in \mathbb{Z}_{>0}$, μ is eigenvalue of $B^pA^qB^r$ if and only if μ is eigenvalue of $B^{p+1}A^qB^{r-1}$.*

Proof. Without loss of generality we may assume B is diagonal, since

$$\begin{aligned}
&B^pA^qB^rx = \mu x \\
\Longleftrightarrow&(VD_2^pV^*)(UD_1^qU^*)(VD_2^rV^*)x = \mu x \\
\Longleftrightarrow&D_2^p(V^*UD_1^qU^*V)D_2^r(V^*x) = \mu(V^*x) \\
\Longleftrightarrow&D_2^p((U^*V)^*D_1^qU^*V)D_2^ry = \mu y, y = V^*x \neq 0,
\end{aligned}$$

while D_2 and $(U^*V)^*D_1^qU^*V$ are positive semidefinite. We first prove the 'only if' part. Let B be diagonal such that first k diagonal entries are nonzero while all the rest are zero, *i.e.*, let

$$(B)_{ii} = \begin{cases} d_i > 0 & \text{if } i \le k \\ 0 & \text{if } i > k \end{cases}.$$

We also define diagonal matrix B' as

$$(B')_{ii} = \begin{cases} d_i^{-1} > 0 & \text{if } i \le k \\ 0 & \text{if } i > k \end{cases}.$$

Note that $(B\mid_{\mathrm{Range}(B)})^{-1} = B'\mid_{\mathrm{Range}(B)}$. If nonzero vector x satisfies $B^pA^qB^rx = \mu x$, since $\mu x \in \mathrm{Range}(B)$ one can conclude that last $(n-k)$ entries of x are all zero, and there exists vector $y \in \mathrm{Range}(B)\setminus\{0\}$ such that $x = B'y$. Then

$$\begin{aligned}
&B^pA^qB^rx = \mu x \\
\Longleftrightarrow&B^pA^qB^rB'y = \mu B'y \\
\Longrightarrow&B^{p+1}A^qB^{r-1}(BB')y = \mu(BB')y \\
\Longrightarrow&B^{p+1}A^qB^{r-1}y = \mu y,
\end{aligned}$$

which is the result we wanted. Proving converse is rather similar:

$$\begin{aligned}
&B^{p+1}A^qB^{r-1}y = \mu y \\
\Longrightarrow&B'B^{p+1}A^qB^{r-1}y = \mu B'y.
\end{aligned}$$

Since $B^{p+1}A^qB^{r-1}y = \mu y$ implies that $y \in \text{Range}(B)$ and there exists $z \in \text{Range}(B)$ such that $z = By$ and we get

$$
\begin{aligned}
&B'B^{p+1}A^qB^{r-1}y = \mu B'y \\
\Longrightarrow &B'B^{p+1}A^qB^{r-1}Bz = \mu B'Bz \\
\Longrightarrow &B^pA^qB^rz = \mu z \ (\because (B\mid_{\text{Range}(B)})^{-1} = B'\mid_{\text{Range}(B)}),
\end{aligned}
$$

which completes the proof. $\square$

As an immediate consequence for any $m \geq 1$, eigenvalues of

$$A^mB^m = (\sqrt{A})^{2m}(\sqrt{B})^{2m} = (\sqrt{A})^{m+m}(\sqrt{B})^{2m}(\sqrt{A})^{m-m}$$

are all real and nonnegative, since $(\sqrt{A})^m(\sqrt{B})^{2m}(\sqrt{A})^m = ((\sqrt{B})^m(\sqrt{A})^m)^*((\sqrt{B})^m(\sqrt{A})^m)$ thus positive semidefinite. Showing that B^mA^m has only real nonnegative eigenvalues can be done in similar sense. Now we prove the case of $m = 1$. Note that whenever μ is eigenvalue of M, $\overline{\mu}$ is eigenvalue of M^*(Perform the triangularization on M and take conjugate). We have following inequalities

$$
\begin{aligned}
\lambda(AB) &\leq \sqrt{\lambda((AB)^*AB)} \\
&= \sqrt{\lambda(BA^2B)} \\
&= \sqrt{\lambda(B^2A^2)} \\
&= \sqrt{\lambda(A^2B^2)} \ (\because A^2B^2 = (B^2A^2)^* \text{ and eigenvalues of } B^2A^2 \text{ are all real })
\end{aligned}
$$

, thus one gets the proof of the case of $m = 1$. Now Assume we have $[\lambda(A^{m-1}B^{m-1})]^{\frac{1}{m-1}} \leq [\lambda(A^mB^m)]^{\frac{1}{m}}$. Let $\|\cdot\|$ be usual matrix norm, which is induced from euclidean norm on $\mathbb{C}^n$. Then following inequalities hold:

$$
\begin{aligned}
\lambda(A^mB^m) &= \lambda((\sqrt{B})^m(\sqrt{A})^{2m}(\sqrt{B})^m) \\
&= \lambda((\sqrt{B})^{m-1}(\sqrt{A})^{2m}(\sqrt{B})^{m+1}) \\
&\leq \|(\sqrt{B})^{m-1}(\sqrt{A})^{2m}(\sqrt{B})^{m+1}\| \\
&\leq \|(\sqrt{B})^{m-1}(\sqrt{A})^{m-1}\|\|(\sqrt{B})^{m+1}(\sqrt{A})^{m+1}\| \\
&= \sqrt{\lambda(((\sqrt{B})^{m-1}(\sqrt{A})^{m-1})^*(\sqrt{B})^{m-1}(\sqrt{A})^{m-1})} \\
&\times \sqrt{\lambda(((\sqrt{B})^{m+1}(\sqrt{A})^{m+1})^*(\sqrt{B})^{m+1}(\sqrt{A})^{m+1})} \\
&= \sqrt{\lambda(A^{m-1}B^{m-1})\lambda(A^{m+1}B^{m+1})}
\end{aligned}
$$

Last two equalities become clear if one performs polar decomposition on

$$(\sqrt{B})^{m-1}(\sqrt{A})^{m-1} \text{ and } (\sqrt{B})^{m+1}(\sqrt{A})^{m+1},$$

and we get

$$
\begin{aligned}
\lambda(A^{m+1}B^{m+1}) &\geq \frac{\lambda(A^mB^m)^2}{\lambda(A^{m-1}B^{m-1})} \\
&\geq \frac{\lambda(A^mB^m)^2}{\lambda(A^mB^m)^{\frac{m-1}{m}}} \\
&= \lambda(A^mB^m)^{\frac{m+1}{m}},
\end{aligned}
$$

which is equivalent to

$$[\lambda(A^mB^m)]^{\frac{1}{m}} \leq [\lambda(A^{m+1}B^{m+1})]^{\frac{1}{m+1}}.$$