2021-03 A placement of rooks on a chessboard

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March 19, 2021

Problem.

Consider an n by n chessboard with white/black squares alternating on every row and every column. In how many ways can one choose k white squares and $n-k$ black squares from this chessboard with no two squares in a row or column.

Solution.

Let's color the square black if row index number and column index number of the square has same parity, white if they have different parity.

If x squares were chosen to have even row index number and odd column index number, $\lceil n/2 \rceil - x$ squares should be chosen to have odd row index number and odd column index number. There are $\lceil n/2 \rceil$ even numbers less than or equal to n, and every number less than n should be chosen as column index number exactly once.

Similarly, there are $\lfloor n/2 \rfloor$ odd numbers less than or equal to n, so $\lfloor n/2 \rfloor - x$ squares should be chosen to have even row index number and even column index number, and x squares should be chosen to have odd row index number and even column index number. There are $2x = k$ white squares chosen. k cannot be odd, so the number of cases is 0 if k is odd.

To count the number of cases, first we choose set of odd row indices where square with even column index number is chosen. x indices out of $\lceil n/2 \rceil$ should be chosen, so the number of cases is $\binom{\lceil n/2 \rceil}{x}$. Secondly, set of even row indices where square with odd column index number is chosen. x indices out of $\lfloor n/2 \rfloor$ should be chosen, so the number of cases is $\binom{\lfloor n/2 \rfloor}{x}$. Finally, we can choose permutation for even column indices and odd column indices freely. The number of cases is $\lfloor n/2 \rfloor! \lceil n/2 \rceil!$. Using rule of product, the number of cases choosing squares is $\binom{\lceil n/2 \rceil}{x}\binom{\lfloor n/2 \rfloor}{x} (\lfloor n/2 \rfloor)! (\lceil n/2 \rceil)!$

So the answer is given as follows:

$$\begin{cases} \binom{\lceil n/2 \rceil}{k/2}\binom{\lfloor n/2 \rfloor}{k/2} (\lfloor n/2 \rfloor)! (\lceil n/2 \rceil)! & \text{if } 2 \mid k \\ 0 & \text{if } 2 \nmid k \end{cases}$$