2021-02 Inscribed triangles

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Problem.

Show that for any triangle T and any Jordan curve C in the Euclidean plane, there exists a triangle inscribed in C which is similar to T.

Solution.

If we fix two vertices N, M of a triangle, other vertex L to make triangle ΔNML similar to T (with same vertices order and does not allow reflection) is unique, and T is continuous subject to N, and M. For some 2-dimensional transformation matrix R, $T = R(N - M) + M$.

Assume we found two triangles $\Delta N_0M_0L_0$, $\Delta N_1M_1L_1$ which are similar to T, such that N_0, N_1, M_0, M_1 is on the curve C, L_0 is in the inside of C, and L_1 is in the outside of the curve C. We can continuously move N_0 and M_0 to N_1 and M_1, L_0 moves continuously to L_1 while crossing C.

To find $\Delta N_0M_0L_0$, take any point P in the inside of C, and find the maximum possible disk centered in C, and do not intersect with the outside of C. such disk will intersect with C with a point N_0. We can draw a triangle ΔN_0ML, inscribed in the disk, which is similar to T. We will also assume $\overline{ML}$ as the longest side of the triangle.

By rotating ΔN_0ML with center N_0 continuously, while either one of M or L will meet with C. (If not, trace of M will form closed curve D, which is in the inside of C, since M is in the inside of C. However N_0, which on the curve C, should be in the inside of D, since N_0 is the center of rotation, which leads to contradiction.) Without lost of generality, let's assume M met with C. M and L can be chosen as M_0 and L_0. N_0 and M_0 lies on C, and L_0 is in the inside of C or lies on C.

To find $\Delta N_1M_1L_1$, let's choose two furthest points of C as N_1 and M_1, and choose L_1 such that $\Delta N_1M_1L_1$ is similar to T. We have set $\overline{ML}$ as the longest side of the triangle, so $\overline{M_1L_1} \geq \overline{N_1M_1}$, and L_1 should be in the outside of C or lie on C.

If L_0 or L_1 lies on C, such triangle $\Delta N_0M_0L_0$ or $\Delta N_1M_1L_1$ will complete the proof. Otherwise, we can continously move $\Delta N_0M_0L_0$ from $\Delta N_1M_1L_1$ to find the triangle which L lies on C.

Technically, let's denote parametrization of C as φ. ($\varphi : [0, 1] \to \mathbb{R}^2$, φ is continuous, $\varphi(0) = \varphi(1)$, and restriction of φ to $[0, 1)$ is injective.) Let's define n_0, n_1, m_0, m_1 to be reals which meets $N_0 = \varphi(n_0), N_1 = \varphi(n_1), M_0 = \varphi(m_0), M_1 = \varphi(m_1)$.

For $t \in [0, 1]$, we can denote move of L using following parametrization: $N_t = \varphi(tn_1 + (1 - t)n_0)$, $M_t = \varphi(tn_1 + (1 - t)n_0)$, $L_t = R(N_t - M_t) + M_t$. L_t is continuous subject to N_t and M_t, also N_t and M_t is subject to t. So L_t is continuous subject to T. This means L_t is parametrization of arc connecting L_0 and L_1. If L_0 is in the inside of C, and L_1 is in the outside of C, which is well-defined by Jordan Curve Theorem, there must be $v \in (0, 1)$ such that L_v is on the curve C. (We can take supremum of t such that L_t is in the inside of the C.) $\Delta N_v M_v L_v$ is triangle which inscribed in C, which is also similar to T. $\square$