2021-01 Single-digit number

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Problem.

Prove that for any given positive integer n, there exists a sequence of the following operations that transforms n to a single-digit number (in decimal representation).

1. multiply a given positive integer by any positive integer.
2. remove all zeros in the decimal representation of a given positive integer.

Solution.

We denote a can be reduced to b, if b can be reached using zero or more finite operation starting from a. We denote this as $a \to b$. It is trivial that $a \to b$ and $b \to c$ implies $a \to c$. Also, if b is multiple of a, then $a \to b$.

We will show every integer can be reduced to 9.

First, we will only consider about some special form of number, $(10^9k - 1)/9 = \underbrace{11\cdots1}_{9k \text{ ones}}$.

Lemma 1. *Every integer can be reduced to a number form of* $(10^{9k} - 1)/9$.

proof.

For $n = 2^a5^bm$ where $\gcd(m, 10) = 1$, we can multiply 2^b5^a to n using first operation, we can get $m \times 10^{a+b}$, and by removing leading $a + b$ zeroes using second operation, we can get m but zeroes removed in decimal representation.

By Euler's theorem $10^{\phi(m)} \equiv 1 \pmod{m}$, thus $\sum_{i=0}^{9m-1} 10^{i\phi(m)} \equiv \sum_{i=0}^{9m-1} 1 = 9m \equiv 0 \pmod{m}$. Thus, $m \to \sum_{i=0}^{9m-1} 10^{i\phi(m)}$, and by using second operation to this integer, $\sum_{i=0}^{9m-1} 10^{i\phi(m)} \to \sum_{i=0}^{9m-1} 10^i = (10^{9m} - 1)/9$. Thus all integers can be reduced to a number form of $(10^{9k} - 1)/9$. □

We only need to prove that every integer form of $(10^{9k} - 1)/9$ can be reduced to 9. By observing $11 \to 10208 = 11 \times 928 \to 128 = 2^7 \to 10000000 = 2^7 \times 5^7 \to 1$, we will expand 11 to 10208 in $(10^{2n} - 1)/9$, remove all zeroes, and multiply 5^7, thus halve the number of ones.

Lemma 2. $\forall n \in Z^+, (10^{2n}-1)/9 \to (10^n-1)/9$.

proof.
$\sum_{i=0}^{n-1}(10208 \times 10^{(6n+2)i}) \equiv \sum_{i=0}^{n-1}(10208 \times 10^{2i}) = 928\sum_{i=0}^{n-1}(11 \times 10^{2i}) = 928 \times \sum_{i=0}^{2n-1} 10^i = 928 \times (10^{2n}-1)/9 \equiv 0 \pmod{(10^{2n}-1)/9}$. Thus $(10^{2n}-1)/9 \to \sum_{i=0}^{n-1}(10208 \times 10^{(6n+2)i})$.

Applying second operation gives $\sum_{i=0}^{n-1}(10208 \times 10^{(6n+2)i}) \to \sum_{i=0}^{n-1}(128 \times 10^{3i})$. By multiplying 5^7, $\sum_{i=0}^{n-1}(128 \times 10^{3i}) \to \sum_{i=0}^{n-1} 10^{3i+7}$, and applying second operation gives $\sum_{i=0}^{n-1} 10^{3i+7} \to \sum_{i=0}^{n-1} 10^i = (10^n-1)/9$. □

Now, we should handle odd numbers. Odd numbers can be handled using following similar lemma. Here, 11 is represented as 11 ones in decimal representation.

Lemma 3. $\forall 2 \le n \in Z^+, (10^n-1)/9 \to (10^{n+9}-1)/9$.

proof.
$(\sum_{i=0}^{n-3} 10^i) + 10^{n-2} \times (\sum_{i=0}^{10} 10^{in}) \equiv (\sum_{i=0}^{n-3} 10^i) + 10^{n-2} \times (\sum_{i=0}^{10} 1) = \sum_{i=0}^{n-1} 10^i \equiv 0 \pmod{(10^n-1)/9}$. Thus $(10^n-1)/9 \to (\sum_{i=0}^{n-3} 10^i) + 10^{n-2} \times (\sum_{i=0}^{10} 10^{in})$.

By applying second operation (there are $n+9$ ones in decimal representation), $(\sum_{i=0}^{n-3} 10^i) + 10^{n-2} \times (\sum_{i=0}^{10} 10^{in}) \to \sum_{i=0}^{n+8} 10^i = (10^{n+9}-1)/9$. □

By using Lemma 2 and 3, large numbers can be reduced to $(10^9-1)/9 = 111111111$ easily.

Lemma 4. $\forall n \in Z^+, (10^{9n}-1)/9 \to (10^9-1)/9$.

proof.
Let's use Strong Mathematical Induction on n.
base step. $n = 1$.
If we apply zero operations, $(10^9-1)/9 \to (10^9-1)/9$.
inductive step. $n = 1, \cdots, m-1 \implies n = m \ (m \ge 2)$

- If $m = 2k$ is even, $(10^{18k}-1)/9 \to (10^{9k}-1)/9$ using lemma 2. $(18k > 9k)$
- If $m = 2k+1$ is odd, $(10^{18k+9}-1)/9 \to (10^{18k+18}-1)/9 \to (10^{9k+9}-1)/9$ using lemma 2 and 3. $(18k+9 > 9k+9)$

By induction hypothesis, $(10^{9m}-1)/9$ can be reduced to $(10^9-1)/9$ regardless of m is even or odd. □

Now we need to prove that 111111111 can be reduced to 9. This takes some calculation.

Lemma 5. $111111111 \to 9$

proof.
$X = 60000040000000800501000080000003004000060000030401000351424$,
$111111111 \to X \to 648518346341351424 = 9 \times 2^{56} \to 9 \times 10^{56} \to 9$. $\square$

This number is found using following steps.

First, we want to make a number form of 9×10^k, so we could use second operation and remove zeroes, finally reduce to 9. Numbers form of 9×2^k and 9×5^k are divisor of 9×10^k which might not contain zeroes in their decimal representation, so we might want to pass by those numbers if possible.

If the digits appear in decimal representation can be partitioned as 9 multisets where their sums are equal, then we can insert zeroes arbitrarily and make the number multiple of $(10^9 - 1)/9$. We can put the digits in i'th multiset in $(9k + i)$'th digit. Sum of digits of 9×2^{56} is 72, we should partition digits into 9 multisets where their sums are 8, and luckily, the number does not contain 0 or 9 as digit.

If we write the X to fit in 7×9 grid, sum of each column is equal. This can verify that the X is multiple of $(10^9 - 1)/9$, not by manually dividing it.

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     6000
004000000
800501000
080000003
004000060
000030401
000351424
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X written in 7×9 grid.

Finally, we can prove any positive integer can be reduced to 9.

Theorem 6. $\forall n \in Z^+, n \to 9$.

By lemma 1, $n \to (10^{9m} - 1)/9$ for some m. By lemma 4 and 5, $(10^{9m} - 1)/9 \to (10^9 - 1)/9 \to 9$. Thus $\forall n \in Z^+, n \to 9$. $\square$