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Sunghun Ko

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Problem. *Prove that either R or $F_m - R$ is a Fibonacci number.*

We begin with mentioning two properties of Fibonacci sequence.

Lemma 1. *For every $n, m > 0$, $F_{n+m} = F_m F_{n+1} + F_{m-1} F_n$.*

Proof.

$$\begin{aligned}
F_{n+m} &= F_{n+m-1} + F_{n+m-2} \\
&= (F_{n+m-2} + F_{n-m-3}) + F_{n+m-2} \\
&= 2F_{n+m-2} + F_{n+m-3} \\
&= 2(F_{n+m-3} + F_{n+m-4}) + F_{n+m-3} \\
&= 3F_{n+m-3} + 2F_{n+m-4} \\
&= \cdots \\
&= F_m F_{n+1} + F_{m-1} F_n
\end{aligned}$$

□

Lemma 2. *For every $m > n \geq 0$, $F_{m-n} = (-1)^{n-2} F_{n-1} F_m + (-1)^{n-1} F_n F_{m-1}$.*

Proof.

$$\begin{aligned}
F_{m-n} &= F_{m-n+2} - F_{m-n+1} \\
&= (-1)^{1-1} F_1 F_{m-n+2} + (-1)^1 F_2 F_{m-n+1} \\
&= (-1)^{1-1} F_1 F_{m+n-2} - (-1)^1 F_2 (F_{m-n+3} - F_{m-n+2}) \\
&= (-1)^{2-1} F_2 F_{m-n+3} + (-1)^2 F_3 F_{m-n+2} \\
&= (-1)^{2-1} F_2 F_{m-n+3} + (-1)^2 F_3 (F_{m-n+4} - F_{m-n+3}) \\
&= \cdots \\
&= (-1)^{n-2} F_{n-1} F_{m-n+(n-1)+1} + (-1)^{n-1} F_n F_{m-n+(n-1)} \\
&= (-1)^{n-2} F_{n-1} F_m + (-1)^{n-1} F_n F_{m-1}
\end{aligned}$$

□

Now, let's assume $n = km + r$ for some integer k and $0 \leq r < m$. Then, by lemma 1, we have

$$\begin{aligned}
F_n &= F_{km+r} \\
&= F_m F_{(k-1)m+r+1} + F_{m-1} F_{(k-1)m+r} \\
&\equiv F_{m-1} F_{(k-1)m+r} \pmod{F_m} \\
&\equiv F_{m-1}^2 F_{(k-2)m+r} \pmod{F_m} \\
&\equiv \cdots \\
&\equiv F_{m-1}^k F_r \pmod{F_m}
\end{aligned}$$

Also, by lemma 2, one can deduce that

$$\begin{aligned}
1 = F_{m-(m-2)} &= (-1)^{m-2-2} F_{m-3} F_m + (-1)^{m-2-1} F_{m-2} F_{m-1} \\
&\equiv (-1)^{m-3} (F_m - F_{m-1}) F_{m-1} \pmod{F_m} \\
&\equiv (-1)^{m-2} F_{m-1}^2 \pmod{F_m}
\end{aligned}$$

Thus $F_{m-1}^2 \equiv (-1)^m \pmod{F_m}$ and we get

$$\begin{aligned}
F_n &\equiv F_{m-1}^k F_r \pmod{F_m} \\
&\equiv \begin{cases} ((-1)^m)^{\lfloor k/2 \rfloor} F_{m-1} F_r \pmod{F_m} & \text{,if } k \text{ is odd} \\ F_r \pmod{F_m} & \text{,if k is even} \end{cases}
\end{aligned}$$

Again, by lemma 2,

$$F_{m-r} = (-1)^{r-2} F_{r-1} F_m + (-1)^{r-1} F_r F_{m-1} \equiv (-1)^{r-1} F_r F_{m-1} \pmod{F_m}$$

and we can conclude that $F_n \equiv F_r$ or $\pm F_{m-r}$, which completes the proof.