Problem of the Week

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20160520 Junho Lee

Problem. Let S be the unit sphere in $\mathbb{R}^n$, centered at the origin, and $P_1P_2\ldots P_{n+1}$ a regular simplex inscribed in S. Prove that for a point P inside S,

$$\sum_{i=1}^{n+1} PP_i^4$$

depends only on the distance OP (and n).

Proof.

Claim. Centroid of the simplex is origin O.

Let the Centroid of the simplex be G. For each distinct i and j, $P_iP_k=P_jP_k$ for any k distinct from i, j, so P_k lie on perpendicular bisector of P_iP_j. Thus, reflection by the perpendicular bisector (plane) of P_iP_j act on set $\{P_1, P_2, \ldots, P_{n+1}\}$ as $(P_i\ P_j)$. Since centroid is not affected by changing indices, reflection by the bisector of P_iP_j fixes G. Therefore, G is on bisector of P_iP_j, so $GP_i=GP_j$. Because this holds for arbitrary i and j, G is equidistant from P_i, i.e. it is center of the n-circumsphere. Since $(n+1)$ points uniquely decides its n-circumsphere and the simplex is inscribed in S, G is center of S. □

Let r be the distance OP, and θ_i be angle $\angle POP_i$. Then,

$$PP_i^2=\left|\vec{OP}_i-\vec{OP}\right|^2=\left|\vec{OP}_i\right|^2+\left|\vec{OP}\right|^2-2\vec{OP}_i\circ\vec{OP}=1+r^2-2r\cos\theta_i$$

$$PP_i^4=(1+r^2)^2-4r(1+r^2)\cos\theta_i+4r^2\cos^2\theta_i\ .$$

Enough to show that $\sum_{i=1}^{n+1}\cos\theta_i$, $\sum_{i=1}^{n+1}\cos^2(\theta_i)$ depends only on n.

Let $\alpha_i=\vec{OP}_i$, $v=\vec{OP}$. Then, $\cos\theta_i=1/r\ \alpha_i\circ v$.

Since O is centroid of the simplex, we can obtain the following.

$$\sum_{i=1}^{n+1}\alpha_i=0\ \Rightarrow\ \sum_{i=1}^{n+1}\alpha_i\circ v=0$$

As r is fixed, this implies $\sum_{i=1}^{n+1}\cos\theta_i=0$.

Observe that regular (n+1)-simplex spans on n-dimension. Clearly, α_i-α_{n+1} for $1\le i\le n$ is independent, and these spans the whole n-dimensional space.

Hence, we can parameterize v, i.e. $v=\sum_{i=1}^{n}x_i(\alpha_i-\alpha_{n+1})$. Set $x_{n+1}=-\sum_{i=1}^{n}x_i$ so that $v=\sum_{i=1}^{n+1}x_i\alpha_i$.

Also, triangle OP_iP_j is congruent for any i, j because P_iP_j is constant and OP_i=1. Using this, we obtain

$$\alpha_i \circ \alpha_j = \cos\phi \ (i \neq j), \ \alpha_i \circ \alpha_i = 1 \ .$$

Because $\sum_{i=1}^{n+1} \alpha_1 \circ \alpha_i = 0$ holds, $\cos\phi = -1/n$. Now, we can calculate the following.

$$\sum_{i=1}^{n+1} (\alpha_i \circ v)^2 = \sum_{i=1}^{n+1} [\sum_{j=1}^{n+1} x_j \alpha_i \circ \alpha_j]^2 = \sum_{i=1}^{n+1} [(1+1/n) x_i - 1/n \sum_{j=1}^{n+1} x_j]^2 = \frac{(n+1)^2}{n^2} \sum_{i=1}^{n+1} x_i^2$$

Where the last equality holds as sum of x_i is 0 by definition of x_{n+1}.

Plus, we can calculate the following as well.

$$r^2 = v \circ v = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (x_i \alpha_i) \circ (x_j \alpha_j) = \sum_{i=1}^{n+1} x_i \sum_{j=1}^{n+1} x_j \alpha_i \circ \alpha_j = \sum_{i=1}^{n+1} x_i [(1+1/n) x_i - 1/n \sum_{j=1}^{n+1} x_j] = \frac{(n+1)}{n} \sum_{i=1}^{n+1} x_i^2$$

Therefore, by comparing the two we get

$$\sum_{i=1}^{n+1} (\alpha_i \circ v)^2 = \frac{(n+1)}{n} r^2 \quad ,$$

which implies that $\sum_{i=1}^{n+1} \cos^2 \theta_i = (n+1)/n$, depending only on n. □