Solution for POW2020-19

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Question.

Let \( n \) be a positive integer. Determine all continuous functions \( f : [0, 1] \to \mathbb{R} \) such that

\[
\sum_{i=1}^{n} f(x_i) = 1
\]

for all \( x_1, x_2, \ldots, x_n \in [0, 1] \) satisfying \( x_1 + x_2 + \cdots + x_n = 1 \).

Solution.

Let us say \( f \) is \( n \)-nice if \( f \) satisfies the given property with positive integer \( n \).

[0] If \( f(0) = 0 \) and \( f \) is \( n \)-nice, then \( f \) is also \( m \)-nice for \( 1 \leq m < n \).

Suppose that \( f(0) = 0 \), \( f \) is \( n \)-nice, and \( 1 \leq m < n \). Let \( y_1, y_2, \ldots, y_m \in [0, 1] \) satisfying \( y_1 + y_2 + \cdots + y_m = 1 \). Then by letting \( x_1 = y_1, x_2 = y_2, \ldots, x_m = y_m \) and \( x_{m+1} = \cdots = x_n = 0 \), we have \( x_1 + x_2 + \cdots + x_n = 1 \), therefore \( f(y_1) + f(y_2) + \cdots + f(y_m) = f(x_1) + f(x_2) + \cdots + f(x_n) = 1 \) holds. So we can conclude that \( f \) is \( m \)-nice.

[1] The case when \( n = 1 \)

If \( n = 1 \) and \( f \) is \( 1 \)-nice, then clearly \( f \) can be all continuous functions, with \( f(1) = 1 \).

[2] The case when \( n = 2 \)

If \( n = 2 \), then \( f \) is \( 2 \)-nice if and only if it satisfies the following:

\[
f(x) + f(1-x) = 1 \quad \forall x \in [0, 1]
\]

Let \( g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R} \) as \( g(x) = f(x + \frac{1}{2}) - \frac{1}{2} \). Then we have that

\[
f(x) + f(1-x) = 1 \quad \forall x \in [0, 1] \iff g(-x) = f(-x + \frac{1}{2}) - \frac{1}{2} - (1 - f(-x + \frac{1}{2})) = \frac{1}{2} - f(x + \frac{1}{2}) = -g(x) \quad \forall x \in [-\frac{1}{2}, \frac{1}{2}]
\]

Therefore (2) is equivalent with that \( g \) is an odd function, and \( f(x) = g(x - \frac{1}{2}) + \frac{1}{2} \) holds.

\[
\therefore f(x) + f(1-x) = 1 \quad \forall x \in [0, 1] \iff f(x) = g(x - \frac{1}{2}) + \frac{1}{2} \quad \text{for odd function } g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}
\]

[3] The case when \( n \geq 3 \) and \( f(0) = 0 \)

Suppose that \( f \) is \( n \)-nice, and \( n \geq 3 \), \( f(0) = 0 \). In this case, I’ll prove that \( f(x) = x \quad \forall x \in [0, 1] \). Suppose that \( f(c) \neq c \) for some \( c \in [0, 1] \). Let \( h(x) = |f(x) - x| \) for \( x \in [0, 1] \). Then \( h \) is continuous function, and \( h \) is not identically 0. By [0], we have \( f \) is \( 2 \)-nice, so \( f \) satisfies the equation (2) and \( h(1-x) = |f(1-x) - (1-x)| = |(1 - f(x)) - (1 - x)| = |x - f(x)| = h(x) \) holds for all \( x \in [0, 1] \).

From that \( h \) is nonnegative continuous function which is not identically 0, we can find \( c \in [0, 1] \) which attains maximum value of \( h \). Also, from that \( h(1-x) = h(x) \quad \forall x \in [0, 1] \), we may assume that \( c \in [0, \frac{1}{2}] \). Note that \( h(c) > 0 \).
Again, by [0], we have $f(c) + f(c) + f(1 - 2c) = 1$ holds as $c + c + (1 - 2c) = 1$. Therefore, we have
\[
f(c) + f(c) + f(1 - 2c) = 1 = c + c + (1 - 2c),\]
so
\[
(f(1 - 2c) - (1 - 2c)) = -2(f(c) - c), \quad h(1 - 2c) = 2h(c) > h(c)
\]
This contradicts to the maximality of $h(c)$, so we obtain that $f(x) = x \forall x \in [0, 1]$.

[4] The case when $n \geq 3$

Suppose that $f$ is $n$-nice, and $n \geq 3$. Take $\hat{f}(x) = f(x) - k + nkx$, where $k = f(0)$. Then for given $x_1, x_2, \ldots, x_n \in [0, 1]$ satisfying $x_1 + x_2 + \cdots + x_n = 1$, $\hat{f}(x_1) + \hat{f}(x_2) + \cdots + \hat{f}(x_n) = (f(x_1) + \cdots + f(x_n)) - nk + nk(x_1 + x_2 + \cdots + x_n) = 1$, so $\hat{f}$ satisfies the given condition. Moreover, $\hat{f}(0) = f(0) - k = 0$ holds, so by referring [3], we have that $\hat{f}(x) = x \forall x \in [0, 1]$. So, $f(x) - k + nkx = x$ holds and we can conclude that $f(x) = (1 - nk)x + k$, for all $x \in [0, 1]$.

Conversely, suppose $f(x) = (1 - nk)x + k$ holds for some constant $k$. Then for $x_1, x_2, \ldots, x_n \in [0, 1]$ satisfying $x_1 + x_2 + \cdots + x_n = 1$, we have $f(x_1) + f(x_2) + \cdots + f(x_n) = (1 - nk)(x_1 + \cdots + x_n) + nk = 1$. So $f$ is $n$-nice.

So, $f$ is $n$-nice if and only if $f(x) = (1 - nk)x + k$, for some constant $k$.

By summarizing [1], [2], and [4], we have that

\[
f(x) \text{ is } n\text{-nice } \iff \begin{cases} 
  f(x) \text{ is continuous function with } f(1) = 1, \text{ when } n = 1 \\
  f(x) = g(x - \frac{1}{2}) + \frac{1}{2} \text{ for odd continuous function } g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}, \text{ when } n = 2 \\
  f(x) = (1 - nk)x + k \text{ for some constant } k \in \mathbb{R}, \text{ when } n \geq 3
\end{cases}
\]