Solution for POW2020-19

20200129 Yuil Kim

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Question.

Let n be a positive integer. Determine all continuous functions $f : [0,1] \to \mathbb{R}$ such that

$$f(x_1) + f(x_2) + \cdots + f(x_n) = 1 \tag{1}$$

for all $x_1, x_2, \ldots, x_n \in [0,1]$ satisfying $x_1 + x_2 + \cdots + x_n = 1$.

Solution.

Let us say f is n-nice if f satisfies the given property with positive integer n.

[0] If $f(0) = 0$ and f is n-nice, then f is also m-nice for $1 \le m < n$.

Suppose that $f(0) = 0$, f is n-nice, and $1 \le m < n$. Let $y_1, y_2, \ldots, y_m \in [0,1]$ satisfying $y_1 + y_2 + \cdots + y_m = 1$. Then by letting $x_1 = y_1, x_2 = y_2, \ldots, x_m = y_m$ and $x_{m+1} = \cdots = x_n = 0$, we have $x_1 + x_2 + \cdots + x_n = 1$, therefore $f(y_1) + f(y_2) + \cdots + f(y_m) = f(x_1) + f(x_2) + \cdots + f(x_n) = 1$ holds. So we can conclude that f is m-nice.

[1] The case when $n = 1$

If $n = 1$ and f is 1-nice, then clearly f can be all continuous functions, with $f(1) = 1$.

[2] The case when $n = 2$

If $n = 2$, then f is 2-nice if and only if it satisfies the following:

$$f(x) + f(1-x) = 1 \ \forall x \in [0,1] \tag{2}$$

Let $g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ as $g(x) = f(x + \frac{1}{2}) - \frac{1}{2}$. Then we have that

$$f(x) + f(1-x) = 1 \ \forall x \in [0,1] \iff$$
$$g(-x) = f(-x + \frac{1}{2}) - \frac{1}{2} = \frac{1}{2} - (1 - f(-x + \frac{1}{2})) = \frac{1}{2} - f(x + \frac{1}{2}) = -g(x) \ \forall x \in [-\frac{1}{2}, \frac{1}{2}]$$

Therefore (2) is equivalent with that g is an odd function, and $f(x) = g(x - \frac{1}{2}) + \frac{1}{2}$ holds.

$$\therefore f(x) + f(1-x) = 1 \ \forall x \in [0,1] \iff f(x) = g(x - \frac{1}{2}) + \frac{1}{2} \text{ for odd function } g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$$

[3] The case when $n \ge 3$ and $f(0) = 0$

Suppose that f is n-nice, and $n \ge 3$, $f(0) = 0$. In this case, I'll prove that $f(x) = x \ \forall x \in [0,1]$. Suppose that $f(c) \ne c$ for some $c \in [0,1]$. Let $h(x) = |f(x) - x|$ for $x \in [0,1]$. Then h is continuous function, and h is not identically 0. By [0], we have f is 2-nice, so f satisfies the equation (2) and $h(1-x) = |f(1-x) - (1-x)| = |(1 - f(x)) - (1-x)| = |x - f(x)| = h(x)$ holds for all $x \in [0,1]$.

From that h is nonnegative continuous function which is not identically 0, we can find $c \in [0,1]$ which attains maximum value of h. Also, from that $h(1-x) = h(x) \ \forall x \in [0,1]$, we may assume that $c \in [0, \frac{1}{2}]$. Note that $h(c) > 0$.

Again, by [0], we have f is 3-nice, so $f(c)+f(c)+f(1-2c)=1$ holds as $c+c+(1-2c)=1$. Therefore, we have

$$f(c)+f(c)+f(1-2c)=1=c+c+(1-2c)\ \text{, so}$$
$$(f(1-2c)-(1-2c))=-2(f(c)-c)\ ,\ h(1-2c)=2h(c)>h(c)$$

This contradicts to the maximality of $h(c)$, so we obtain that $f(x)=x\ \forall x\in[0,1]$.

[4] The case when $n\geq 3$

Suppose that f is n-nice, and $n\geq 3$. Take $\hat{f}(x)=f(x)-k+nkx$, where $k=f(0)$. Then for given x_1, x_2, $\ldots$, $x_n\in[0,1]$ satisfying $x_1+x_2+\cdots+x_n=1$, $\hat{f}(x_1)+\hat{f}(x_2)+\cdots+\hat{f}(x_n)=(f(x_1)+\cdots+f(x_n))-nk+nk(x_1+x_2+\cdots+x_n)=1$, so $\hat{f}$ satisfies the given condition. Moreover, $\hat{f}(0)=f(0)-k=0$ holds, so by referring [3], we have that $\hat{f}(x)=x\ \forall x\in[0,1]$. So, $f(x)-k+nkx=x$ holds and we can conclude that $f(x)=(1-nk)x+k$, for all $x\in[0,1]$.

Conversely, suppose $f(x)=(1-nk)x+k$ holds for some constant k. Then for $x_1,x_2,\ldots,x_n\in[0,1]$ satisfying $x_1+x_2+\cdots+x_n=1$, we have $f(x_1)+f(x_2)+\cdots+f(x_n)=(1-nk)(x_1+\cdots+x_n)+nk=1$. So f is n-nice.

So, f is n-nice if and only if $f(x)=(1-nk)x+k$, for some constant k.

By summarizing [1], [2], and [4], we have that

$$f(x)\text{ is }n\text{-nice}\iff\begin{cases}f(x)\text{ is continuous function with }f(1)=1\text{, when }n=1\\ f(x)=g(x-\frac{1}{2})+\frac{1}{2}\text{ for odd continuous function }g:[-\frac{1}{2},\frac{1}{2}]\to\mathbb{R}\text{, when }n=2\\ f(x)=(1-nk)x+k\text{ for some constant }k\in\mathbb{R}\text{, when }n\geq 3\end{cases}$$