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First, I will show that This process should be terminated and then show that this process should be terminated in at most $F_{n+1} - 1$ steps, where F_n is fibonacci number.

Let $S = \{x_1, x_2, \cdots, x_n\}$ denotes the initial state of the deck where x_i denotes the ith cards from the top.If $n = 1$, it is trivial. Assume that the process terminates for all integer less than n. If $x_n = n$, then this card cannot be moved since any other number is smaller than n. Thus, in this case it is same with the process with $n - 1$ cards so the process will be terminated by induction hyphothesis. Suppose $x_n = k \neq= n$. If n is not placed on the top during the process, k cannot be moved from the bottom, so it is same with the proecess with $n - 1$ cards and thus it will be terminated. Otherwise, if n is placed on the top, k and n exchanges their position and remaining process will be same with the process with $n-1$ cards. Thus the process must be terminated for all n.

Let $f(n, k)$ be the maximum number of shuffle of the process with the deck, which has k cards of 1 and distinct $n - k$ cards in $\{2, 3, \cdots, n\}$. Then trivially $f(1, 1) = 0$, $\forall m \geq n$ $f(n, m) = 0$. I will show that $f(n, k) \leq F_{n-k+2} - 1$ by using induction. Let S be arbitrary initial state of this deck. Assume the deck contains n. Suppose n is placed on the bottom in S. Then by similar argument used in above, the maximum shuffle number in this case is $f(n - 1, k) \leq F_{n-k+1} - 1 \leq F_{n-k+2} - 1$. If the bottom of S is 1, the maximum shuffle number in this case is achieved when n is placed on the top after $f(n - 1, k)$(by regarding n as 1) shuffle, and then shuffle once again to exchange the position between n and 1. The total shuffle is $f(n - 1, k) + 1 \leq F_{n-k+1} \leq F_{n-k+2} - 1$. Thus, we may assume that the bottom of the S is neither 1 nor n. Then the maximum shuffle number in this case is achieved when n is placed on the top after $f(n - 1, k + 1)$ (by regarding n as 1) shuffles, shuffle once again to place n to the bottom, and then shuffle until the process terminates. The maximum number of shuffle required in the last step is $f(n-1, k)$. Then the total number of shuffle is $f(n-1, k+1) + f(n-1, k) + 1 \leq F_{n-k} + F_{n-k+1} - 1 = F_{n-k+2} - 1$. If S does not contains n, the card in the bottom cannot be moved. Thus, in this case the maximum number of shuffle is $f(n - 1, k)$ which is obviously smaller than the previous cases. $\therefore f(n, 1) \leq F_{n+1} - 1$, which corresponds to our problem.

Then

$$F_{n+1} - 1 = \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}} - 1 \leq \frac{\phi^{n+1}}{\sqrt{5}} \leq \phi^n \leq (1.7)^n$$

where $\phi = 1.618\cdots$ and $\psi = -0.618\cdots$.

Below results are the maximum number of shuffle with n cards and initial state of the deck when the maxima is acheived.

If $n = 4$, the maximum number of shuffle is 4, and $(2413), (3142)$
If $n = 5$, the maximum number of shuffle is 7, and (31452)
If $n = 6$, the maximum number of shuffle is 10, and $(365142), (456213), (415263), (564132)$
If $n = 7$, the maximum number of shuffle is 16, and $(3146752), (47612153)$
If $n = 8$, the maximum number of shuffle is 22, and (61578324)
If $n = 9$, the maximum number of shuffle is 30, and (615972834)
If $n = 10$, the maximum number of shuffle is 38, and $(5, 9, 1, 8, 6, 2, 10, 4, 7, 3)$
If $n = 11$, the maximum number of shuffle is 51, and $(4, 9, 11, 6, 10, 7, 8, 2, 1, 3, 5)$
If $n = 12$, the maximum number of shuffle is 65 and $(2, 6, 1, 10, 11, 8, 12, 3, 4, 7, 9, 5)$