Solution for POW2020-17

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Question.

Prove or disprove that a surjective homomorphism from a finitely generated abelian group to itself is an isomorphism.

Solution.

I'll prove the statement. Let us say that group G is 'nice' if every endomorphism from G to itself is an automorphism of G. Our goal is to show that every finitely generated abelian group is nice. I'll use following Lemma.

Lemma. The group $\mathbb{Z}^n$ is nice, for every $n \geq 1$.

Assume that there is an endomorphism ϕ from $\mathbb{Z}^n$ to itself. Then we can find $\mathbf{v}_i = (k_{i1}, k_{i2}, \ldots, k_{in}) \in \mathbb{Z}^n$ satisfying $\phi(\mathbf{v}_i) = \mathbf{e}_i$ where $\mathbf{e}_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{in})$. ($\delta_{ij}$ denotes Kronecker Delta.)

(0) If $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{Z}^n$ and $c \in \mathbb{Z}$, then $\phi(c\mathbf{u}) = c\phi(\mathbf{u})$ holds.

If $c = 0$, then clearly $\phi(c\mathbf{u}) = (0, 0, \ldots, 0) = c\phi(\mathbf{u})$ holds. And if $\phi(c\mathbf{u}) = c\phi(\mathbf{u})$, then it implies $\phi((c+1)\mathbf{u}) = \phi(c\mathbf{u}) + \phi(\mathbf{u}) = c\phi(\mathbf{u}) + \phi(\mathbf{u}) = (c+1)\phi(\mathbf{u})$ holds. By induction, the statement holds for all $c \geq 0$.

In addition, $(0, 0, \ldots, 0) = \phi((0, 0, \ldots, 0)) = \phi(c\mathbf{u} + (-c)\mathbf{u}) = \phi(c\mathbf{u}) + \phi((-c)\mathbf{u}) = c\phi(\mathbf{u}) + \phi((-c)\mathbf{u})$ holds for all $c \geq 0$, so $\phi((-c)\mathbf{u}) = (0, 0, \ldots, 0) - c\phi(\mathbf{u}) = (-c)\phi(\mathbf{u})\ \forall c \geq 0$, $\phi(c\mathbf{u}) = c\phi(\mathbf{u})\ \forall c \leq 0$ holds.

Therefore, $\phi(c\mathbf{u}) = c\phi(\mathbf{u})$ holds for all $c \in \mathbb{Z}$ and $\mathbf{u} \in \mathbb{Z}^n$.

(1) If $c_1, c_2, \ldots, c_n \in \mathbb{Z}$ satisfy $\sum_{i=1}^n c_i\mathbf{v}_i = (0, 0, \ldots, 0)$, then $c_i = 0\ \forall i \in \{1, 2, \ldots, n\}$

For given integers $c_1, c_2, \ldots, c_n$, we have

$$\phi\left(\sum_{i=1}^n c_i\mathbf{v}_i\right) = \sum_{i=1}^n \phi(c_i\mathbf{v}_i) = \sum_{i=1}^n c_i\phi(\mathbf{v}_i) = \sum_{i=1}^n c_i\mathbf{e}_i = (c_1, c_2, \ldots, c_n)$$

therefore $\sum_{i=1}^n c_i v_i = (0, 0, \ldots, 0)$ implies $c_i = 0\ \forall i \in \{1, 2, \ldots, n\}$

(2) If $q_1, q_2, \ldots, q_n \in \mathbb{Q}$ satisfy $\sum_{i=1}^n q_i\mathbf{v}_i = (0, 0, \ldots, 0)$, then $q_i = 0\ \forall i \in \{1, 2, \ldots, n\}$

Suppose that $q_1, q_2, \ldots, q_n \in \mathbb{Q}$ satisfy $\sum_{i=1}^n q_i\mathbf{v}_i = (0, 0, \ldots, 0)$. Let $q_i = \frac{c_i}{d_i}$, where c_i, d_i are integers and $d_i \neq 0$. Take $D = \prod_{i=1}^n d_i$ and let $q_i' = Dq_i \in \mathbb{Z}$. From $\sum_{i=1}^n q_i\mathbf{v}_i = (0, 0, \ldots, 0)$, we have

$$\sum_{i=1}^n q_i'\mathbf{v}_i = D\sum_{i=1}^n q_i\mathbf{v}_i = (0, 0, \ldots, 0)$$

and from that $q_i' \in \mathbb{Z}$, we have $q_i' = 0\ \forall i \in \{1, 2, \ldots, n\}$ by (1). Therefore $q_i = 0\ \forall i \in \{1, 2, \ldots, n\}$ holds.

(3) The group $\mathbb{Z}^n$ is nice, for every $n \geq 1$.

Let us regard $\mathbf{v}_i$ as a vector of a vector space $\mathbb{Q}^n$ over a field $\mathbb{Q}$. Then by (2), we have that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent. Since $\dim(\mathbb{Q}^n) = n$, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ forms a basis for $\mathbb{Q}^n$.

Let us define a linear transformation T form $\mathbb{Q}^n$ to itself, as $T(\mathbf{e}_i) = \phi(\mathbf{e}_i)\ \forall i \in \{1, 2, \ldots, n\}$. Then, for given $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{Z}^n$, T satisfies

$$T(\mathbf{u}) = T\left(\sum_{i=1}^{n} u_i \mathbf{e}_i\right) = \sum_{i=1}^{n} u_i T(\mathbf{e}_i) = \sum_{i=1}^{n} u_i \phi(\mathbf{e}_i) = \phi\left(\sum_{i=1}^{n} u_i \mathbf{e}_i\right) = \phi(\mathbf{u})$$

so ϕ is obtained by restricting T into $\mathbb{Z}^n$. So $T(\mathbf{v}_i) = \mathbf{e}_i\ \forall i \in \{1, 2, \ldots, n\}$ holds, and clearly this T is automorphism of $\mathbb{Q}^n$ as a vector space over $\mathbb{Q}$. Therefore T is injective and ϕ is restriction of T into $\mathbb{Z}^n$, so ϕ is injective.

Therefore, ϕ becomes automorphism of $\mathbb{Z}^n$, so $\mathbb{Z}^n$ is nice. This completes the proof of Lemma.

Now, we are ready to prove the statement.

Since fundamental theorem guarantees that every finitely generated abelian group is isomorphic to $\mathbb{Z}^n \times H$ for some finite group H and $n \geq 0$, it is okay to prove the statement for $G = \mathbb{Z}^n \times H$ where H is finite. Assume that there is an endomorphism ψ from G to itself. It's enough to show that ψ is injective.

[1] Let $\psi((0, 0, \ldots, 0, h)) = (a_1, a_2, \ldots, a_n, h')$ for $a_1, a_2, \ldots, a_n \in \mathbb{Z}^n$, $h, h' \in H$. Then $a_i = 0\ \forall i \in \{1, 2, \ldots, n\}$.

Since $|(0, 0, \ldots, 0, h)|$ is finite, $|\psi((0, 0, \ldots, 0, h))|$ must be also finite. Therefore, $a_i = 0\ \forall i \in \{1, 2, \ldots, n\}$ holds.

[2] If $\psi((a_1, a_2, \ldots, a_n, h)) = (b_1, b_2, \ldots, b_n, h')$ for $a_i, b_i \in \mathbb{Z}$ and $h, h' \in H$, define $\pi : \mathbb{Z}^n \to \mathbb{Z}^n$ as $\pi((a_1, a_2, \ldots, a_n)) = (b_1, b_2, \ldots, b_n)$. Then π is well-defined surjective homomorphism.

If $\psi(a_1, a_2, \ldots, a_n, h_1) = (b_1, b_2, \ldots, b_n, h_1')$ and $\psi(a_1, a_2, \ldots, a_n, h_2) = (b_1', b_2', \ldots, b_n', h_2')$, then

$$(b_1', b_2', \ldots, b_n', h_2') - (b_1, b_2, \ldots, b_n, h_1') = \psi((0, 0, \ldots, 0, h_2 - h_1)) = (0, 0, \ldots, 0, h') \text{ by [1]}$$

so $b_i = b_i'\ \forall i \in \{1, 2, \ldots, n\}$ holds, so π is well-defined.

If we let $\pi((a_1, a_2, \ldots, a_n)) = (b_1, b_2, \ldots, b_n)$ and $\pi((a_1', a_2', \ldots, a_n')) = (b_1', b_2', \ldots, b_n')$, then

$$\psi((a_1, a_2, \ldots, a_n, h_a)) = (b_1, b_2, \ldots, b_n, h_b),\ \psi((a_1', a_2', \ldots, a_n', h_a')) = (b_1', b_2', \ldots, b_n', h_b') \text{ for some } h_a, h_b, h_a', h_b' \in H$$

So $\psi(a_1 + a_1', \ldots, a_n + a_n', h_a + h_a') = (b_1 + b_1', \ldots, b_n + b_n', h_b + h_b')$, therefore $\pi(a_1 + a_1', \ldots, a_n + a_n') = (b_1 + b_1', \ldots, b_n + b_n') = \pi(a_1, \ldots, a_n) + \pi(a_1', \ldots, a_n')$ holds. In addition, $\psi(-a_1, -a_2, \ldots, -a_n, -h_a) = (-b_1, -b_2, \ldots, -b_n, -h_b)$, so $\pi(-(a_1, a_2, \ldots, a_n)) = (-b_1, -b_2, \ldots, -b_n) = -\pi(a_1, a_2, \ldots, a_n)$, so π is an group homomorphism.

Let $(b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n$ be given. For fixed $h' \in H$, we can find $(a_1, a_2, \ldots, a_n, h) \in G$ with $\psi((a_1, a_2, \ldots, a_n, h)) = (b_1, b_2, \ldots, b_n, h')$ since ψ is surjective. Then, we have $\pi(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$, so π is surjective.

[3] If $\psi((0, 0, \ldots, 0, h)) = (0, 0, \ldots, h')$ for $h, h' \in H$, define $\phi : H \to H$ as $\phi(h) = h'$. Then ϕ is surjective homomorphism.

Note that π is surjective homomorphism from $\mathbb{Z}^n$ to itself, so it is isomorphism by Lemma. It is clear that ϕ is well-defined.

ϕ is homomorphism, since for $h_1, h_2 \in H$, we have $\phi(h_1 + h_2) = \psi(0, 0, \ldots, 0, h_1) + \psi(0, 0, \ldots, 0, h_2) = \psi(0, 0, \ldots, 0, h_1 + h_2) = \phi(h_1 + h_2)$ and $\phi(-h_1) = \psi(0, 0, \ldots, 0, -h_1) = -\psi(0, 0, \ldots, 0, h_1) = -\phi(h_1)$.

In addition, for given $h' \in H$, since ψ is surjective, we can find $(a_1, a_2, \ldots, a_n, h) \in G$ satisfying $\psi(a_1, a_2, \ldots, a_n, h) = (0, 0, \ldots, 0, h')$. Here, $\pi(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0)$ and π is an isomorphism, $a_1, a_2, \ldots, a_n$ are all 0. Therefore, $\psi(0, 0, \ldots, 0, h) = (0, 0, \ldots, 0, h')$ and $\phi(h) = h'$ holds. Therefore ϕ is surjective.

[4] G is nice.

Since ϕ is surjective function from H to H and $|H|$ is finite, ϕ must be injective and it becomes an isomorphism. If we assume $(a_1, a_2, \ldots, a_n, h) \in \ker \psi$, then we have that $\pi(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0)$ and therefore $a_i = 0\ \forall i \in \{1, 2, \ldots, n\}$. So $\psi(0, 0, \ldots, 0, h) = (0, 0, \ldots, 0, 0_H)$, so $\phi(h) = 0_H$ holds. As ϕ is an isomorphism, $h = 0_H$ holds.

Therefore, $(a_1, a_2, \ldots, a_n, h) \in \ker \psi$ implies $a_i = 0\ \forall i \in \{1, 2, \ldots, n\}$, $h = 0_H$. Therefore $\ker \psi = \{0_G\}$, and ψ is injective, so it becomes isomorphism of G. This shows that G is nice and completes the proof of the statement.