Solution for POW2020-17

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Question.

Prove or disprove that a surjective homomorphism from a finitely generated abelian group to itself is an isomorphism.

Solution.

I’ll prove the statement. Let us say that group $G$ is ’nice’ if every endomorphism from $G$ to itself is an automorphism of $G$. Our goal is to show that every finitely generated abelian group is nice. I’ll use following Lemma.

**Lemma.** The group $\mathbb{Z}^n$ is nice, for every $n \geq 1$.

Assume that there is an endomorphism $\phi$ from $\mathbb{Z}^n$ to itself. Then we can find $v_i = (k_{i1}, k_{i2}, \ldots, k_{in}) \in \mathbb{Z}^n$ satisfying $\phi(v_i) = e_i$ where $e_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{in})$. ($\delta_{ij}$ denotes Kronecker Delta.)

(0) If $u = (u_1, u_2, \ldots, u_n) \in \mathbb{Z}^n$ and $c \in \mathbb{Z}$, then $\phi(cu) = c\phi(u)$ holds.

If $c = 0$, then clearly $\phi(cu) = (0, 0, \ldots, 0) = c\phi(u)$ holds. And if $\phi(cu) = c\phi(u)$, then it implies $\phi((c + 1)u) = \phi(cu) + \phi(u) = c\phi(u) + \phi(u) = (c + 1)\phi(u)$ holds. By induction, the statement holds for all $c \geq 0$.

In addition, $(0, 0, \ldots, 0) = \phi((0, 0, \ldots, 0)) = \phi(cu + (-c)u) = \phi(cu) + \phi((-c)u) = c\phi(u) + \phi((-c)u)$ holds for all $c \geq 0$, so $\phi((-c)u) = (0, 0, \ldots, 0) - c\phi(u) = (-c)\phi(u)$ for all $c \geq 0$, $\phi(cu) = c\phi(u)$ for all $c \leq 0$ holds.

Therefore, $\phi(cu) = c\phi(u)$ holds for all $c \in \mathbb{Z}$ and $u \in \mathbb{Z}^n$.

(1) If $c_1, c_2, \ldots, c_n \in \mathbb{Z}$ satisfy $\sum_{i=1}^{n} c_i v_i = (0, 0, \ldots, 0)$, then $c_i = 0 \forall i \in \{1, 2, \ldots, n\}$

For given integers $c_1, c_2, \ldots, c_n$, we have

$$\sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} \phi(c_i v_i) = \sum_{i=1}^{n} c_i \phi(v_i) = \sum_{i=1}^{n} c_i e_i = (c_1, c_2, \ldots, c_n)$$

therefore $\sum_{i=1}^{n} c_i v_i = (0, 0, \ldots, 0)$ implies $c_i = 0 \forall i \in \{1, 2, \ldots, n\}$

(2) If $q_1, q_2, \ldots, q_n \in \mathbb{Q}$ satisfy $\sum_{i=1}^{n} q_i v_i = (0, 0, \ldots, 0)$, then $q_i = 0 \forall i \in \{1, 2, \ldots, n\}$

Suppose that $q_1, q_2, \ldots, q_n \in \mathbb{Q}$ satisfy $\sum_{i=1}^{n} q_i v_i = (0, 0, \ldots, 0)$. Let $q_i = \frac{c_i}{d_i}$, where $c_i, d_i$ are integers and $d_i \neq 0$.

Take $D = \prod_{i=1}^{n} d_i$ and let $q'_i = D q_i \in \mathbb{Z}$. From $\sum_{i=1}^{n} q_i v_i = (0, 0, \ldots, 0)$, we have

$$\sum_{i=1}^{n} q'_i v_i = D \sum_{i=1}^{n} q_i v_i = (0, 0, \ldots, 0)$$

and from that $q'_i \in \mathbb{Z}$, we have $q'_i = 0 \forall i \in \{1, 2, \ldots, n\}$ by (1). Therefore $q_i = 0 \forall i \in \{1, 2, \ldots, n\}$ holds.

(3) The group $\mathbb{Z}^n$ is nice, for every $n \geq 1$.

Let us regard $v_i$ as a vector of a vector space $\mathbb{Q}^n$ over a field $\mathbb{Q}$. Then by (2), we have that $v_1, v_2, \ldots, v_n$ are linearly independent. Since $\dim(\mathbb{Q}^n) = n$, $\{v_1, v_2, \ldots, v_n\}$ forms a basis for $\mathbb{Q}^n$. 


Let us define a linear transformation $T$ form $\mathbb{Q}^n$ to itself, as $T(e_i) = \phi(e_i)$ $\forall i \in \{1, 2, \ldots, n\}$. Then, for given $u = (u_1, u_2, \ldots, u_n) \in \mathbb{Z}^n$, $T$ satisfies

$$T(u) = T \left( \sum_{i=1}^{n} u_i e_i \right) = \sum_{i=1}^{n} u_i T(e_i) = \sum_{i=1}^{n} u_i \phi(e_i) = \phi \left( \sum_{i=1}^{n} u_i e_i \right) = \phi(u)$$

so $\phi$ is obtained by restricting $T$ into $\mathbb{Z}^n$. So $T(e_i) = e_i$ $\forall i \in \{1, 2, \ldots, n\}$ holds, and clearly this $T$ is automorphism of $\mathbb{Q}^n$ as a vector space over $\mathbb{Q}$. Therefore $T$ is injective and $\phi$ is restriction of $T$ into $\mathbb{Z}^n$, so $\phi$ is injective.

Therefore, $\phi$ becomes automorphism of $\mathbb{Z}^n$, so $\mathbb{Z}^n$ is nice. This completes the proof of Lemma.

Now, we are ready to prove the statement.

Since fundamental theorem guarantees that every finitely generated abelian group is isomorphic to $\mathbb{Z}^n \times H$ for some finite group $H$ and $n \geq 0$, it is okay to prove the statement for $G = \mathbb{Z}^n \times H$ where $H$ is finite. Assume that there is an endomorphism $\psi$ from $G$ to itself. It’s enough to show that $\psi$ is injective.

1. Let $\psi((0, 0, \ldots, 0, h)) = (a_1, a_2, \ldots, a_n, h')$ for $a_1, a_2, \ldots, a_n \in \mathbb{Z}^n$, $h, h' \in H$. Then $a_i = 0$ $\forall i \in \{1, 2, \ldots, n\}$.

Since $\|0, 0, \ldots, 0\|$ is finite, $\|\psi((0, 0, \ldots, 0))\|$ must be also finite. Therefore, $a_i = 0$ $\forall i \in \{1, 2, \ldots, n\}$ holds.

2. If $\psi((a_1, a_2, \ldots, a_n, h)) = (b_1, b_2, \ldots, b_n, h')$ for $a_i, b_i \in \mathbb{Z}$ and $h, h' \in H$, define $\pi : \mathbb{Z}^n \to \mathbb{Z}^n$ as $\pi((a_1, a_2, \ldots, a_n)) = (b_1, b_2, \ldots, b_n)$. Then $\pi$ is well-defined surjective homomorphism.

If $\psi(a_1, a_2, \ldots, a_n, h_1) = (b_1, b_2, \ldots, b_n, h_1')$ and $\psi(a_1, a_2, \ldots, a_n, h_2) = (b_1', b_2', \ldots, b_n', h_2')$, then

$$(b_1', b_2', \ldots, b_n', h_2') - (b_1, b_2, \ldots, b_n, h_1') = \psi((0, 0, \ldots, 0, h_2 - h_1)) = (0, 0, \ldots, 0, h')$$

by [1] so $b_i = b_i'$ $\forall i \in \{1, 2, \ldots, n\}$ holds, so $\pi$ is well-defined.

If we let $\pi((a_1, a_2, \ldots, a_n)) = (b_1, b_2, \ldots, b_n)$ and $\pi((a_1', a_2', \ldots, a_n')) = (b_1', b_2', \ldots, b_n')$, then

$$\psi((a_1, a_2, \ldots, a_n, h_n)) = (b_1, b_2, \ldots, b_n, h_n), \psi((a_1', a_2', \ldots, a_n', h_n')) = (b_1', b_2', \ldots, b_n', h_n')$$

for some $h_n, h_n', h_n'' \in H$.

So $\psi(a_1 + a_1', \ldots, a_n + a_n', h_n + h_n', h_n'' + h_n'') = (b_1 + b_1', \ldots, b_n + b_n', h_n + h_n', h_n'' + h_n'')$, therefore $\pi(a_1 + a_1', \ldots, a_n + a_n', h_n, h_n') = (b_1 + b_1', \ldots, b_n + b_n', h_n, h_n')$.

Hence $\pi(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ holds.

In addition, $\psi(-a_1, -a_2, \ldots, -a_n, -h_n) = (-b_1, -b_2, \ldots, -b_n, -h_n)$, so $\pi$ is an group homomorphism.

Let $(b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n$ be given. For fixed $h' \in H$, we can find $(a_1, a_2, \ldots, a_n, h) \in G$ with $\psi((a_1, a_2, \ldots, a_n, h)) = (b_1, b_2, \ldots, b_n, h')$ since $\psi$ is surjective. Then, we have $\pi(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$, so $\pi$ is surjective.

3. If $\psi((0, 0, \ldots, 0, h)) = (0, 0, \ldots, h')$ for $h, h' \in H$, define $\phi : H \to H$ as $\phi(h) = h'$. Then $\phi$ is surjective homomorphism.

Note that $\pi$ is surjective homomorphism from $\mathbb{Z}^n$ to itself, so it is isomorphism by Lemma. It is clear that $\phi$ is well-defined.

$\phi$ is homomorphism, since for $h_1, h_2 \in H$, we have $\phi(h_1 + h_2) = \psi(0, 0, \ldots, 0, h_1) + \psi(0, 0, \ldots, 0, h_2) = \psi(0, 0, \ldots, 0, h_1 + h_2) = \phi(h_1 + h_2)$ and $\phi(-h) = \psi(0, 0, \ldots, 0, -h) = -\psi(0, 0, \ldots, 0, h) = -\phi(h)$.

In addition, for given $h' \in H$, since $\psi$ is surjective, we can find $(a_1, a_2, \ldots, a_n, h) \in G$ satisfying $\psi((a_1, a_2, \ldots, a_n, h)) = (0, 0, \ldots, 0, h')$. Hence, $\pi(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0)$ and $\pi$ is an isomorphism, $a_1, a_2, \ldots, a_n$ are all 0. Therefore, $\psi(0, 0, \ldots, 0, h) = (0, 0, \ldots, 0, h')$ and $\phi(h) = h'$ holds. Therefore $\phi$ is surjective.

4. $G$ is nice.

Since $\phi$ is surjective function from $H$ to $H$ and $|H|$ is finite, $\phi$ must be injective and it becomes an isomorphism. If we assume $(a_1, a_2, \ldots, a_n, h) \in \ker \psi$, then we have that $\pi(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0)$ and therefore $a_i = 0$ $\forall i \in \{1, 2, \ldots, n\}$. So $\psi(0, 0, \ldots, 0, h) = (0, 0, \ldots, 0, 0_H)$, so $\phi(h) = 0_H$ holds. As $\phi$ is an isomorphism, $h = 0_H$ holds.

Therefore, $(a_1, a_2, \ldots, a_n, h) \in \ker \psi$ implies $a_i = 0$ $\forall i \in \{1, 2, \ldots, n\}$, $h = 0_H$. Therefore $\ker \psi = \{0_G\}$, and $\psi$ is injective, so it becomes isomorphism of $G$. This shows that $G$ is nice and completes the proof of the statement.